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Decisions under risk programming with economic applications

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DECISIONS UNDER RISK PROGRAMMING WITH

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1. INTRODUCTION

Mathematical programming (145, 61) is concerned with the problem of maximizing or minimizing a function of variables that are restricted by a number of constraints. Interest in this problem arose in economics and management sciences, where it was realized that many problems of optimum allocation of scarce resources could be formulated mathematically as programming problems. The introduction of large light-spaced electronic computers, moreover, made it possible in principle to obtain numerical solutions, provided efficient mathematical methods and computational techniques could be developed.

The concept of risk has played a very significant role in the theory of production, resource allocation, and the theory of statistical decisions. This is used here for analyzing the implications of a certain type of probabilistic programming models in a linear and non-linear framework. The plan of the discussion covering this topic will be as follows.

In Chapter 2 a general survey of risk programming is presented; the various methods of risk programming discussed in those sections are limited mostly to probabilistic linear programming of static variety, under simplifying assumptions regarding the statistical distributions of the parameters of the problems which are (A,b,c). This is followed by Chapter

3 which presents the optimization techniques used in this thesis in order to deal with nonlinear deterministic forms produced by the risk programming problems. Here we are using the most powerful numerical techniques which enable us to solve problems that were impossible to solve in the past. Chapter 4 examined an empirical model of production planning and the emphasis here is on evaluating a reliability level for every period of production planning and comparing with chance constrained programming. We also considered an investment situation, where an investor wished to select a portfolio; this original model first formulated by Näslund (90, 89) is solved here in the general nonlinear fashion, extended to the nonlinear reliability programming case and analyzed in its sample distribution aspects. In Chapter 5 we indicated two applications of geometric programming to economic models, in the first case we use generalized polynomial programming applied to the simple stochastic production model, in the second case we applied geometric programming to the posynomials arising in Uzawa's model of economic growth (135).

Finally, a broad summary of all principal results and ideas for future research is presented in Chapter 6.

2. REVIEW OF LITERATURE IN RISK PROGRAMMING

Since the origin of the species, men have been making decisions, and experts have been telling them how they make, or should make decisions. Von Neumann and Morgenstern (91) developed a theory of maximizing the expected utility. In order for their results to be valid, however, their assumption that rational individuals are choosing the right utility function must hold true.

The fundamental problem of production is the optimum allocation of scarce resources between alternative ways of achieving an objective. It can be seen that the objective may be the maximization of the firm's profit or the minimization of costs. Cases exist, however, in which besides profit maximization or cost minimization, the objectives include risk minimization. If the decision-maker is willing to sacrifice profit in exchange for security, the result depends on his behavior.

The difference between uncertainty and risk must be pointed out here. Each term has had distinct meaning in different parts of economic literature. The term "risk" is characterized in a model in which the entire probability distribution of the outcomes has formally been taken into account, whether the character of that distribution is considered subjective or objective. The term "uncertainty" is applied to models in which the above stated conditions are not the case.

It is very important to have a realistic theory explaining how individuals choose among alternate courses of action when the consequences of their actions are not fully known to them. A survey of the literature of approaches to the theory of choices in risk taking situations has been given by Arrow (3).

The probability theory represents the sustained efforts of mathematicians and philosophers to provide a rational basis on which expectations may be derived from past events. Roy (102) stated that there are major objections when one attempts to maximize expected gain or profit. The ordinary man has to consider the possible outcomes of a given course of action on one occasion only, and the average or expected outcome, if this conduct were repeated a large number of times under similar conditions, is irrelevant. Also, the well-known phenomenon of the diversification of resources among a wide range of project or investment situations is not explained.

Mathematical programming under risk has been developed during the last ten years and there exist many published articles in the field. The research, to date, can be divided in three major areas (88) namely:

a) stochastic linear programming

b) linear programming under uncertainty

c) chance constrained programming

All methods start from the linear programming formulation namely minimizing or maximizing a linear function subject to linear constraints. Stochastic linear programming is mainly concerned with studying the statistical distribution of max z. This work has followed two lines namely the so called "passive" and "active" approaches. Programming under uncertainty partitions the problem into two or more stages. First the decision maker selects an initial decision, then the random effects occur. Chance-constrained programming allows for constraint violations a certain proportion of the time. Reliability programming is an extension of chance constrained programming where the probability levels are not preassigned, but optimized in some sense.

2.1. Stochastic Linear Programming

The ordinary linear programming problem can be formulated as follows (35, 57, 60)

$$\text{Max } c'x \qquad [2.1.1]$$

subject to

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned} \qquad [2.1.2]$$

where

c is a row vector with n elements

x is a column vector with n elements

A is a m by n matrix

b is a column vector with m elements.

The problem is to find values for the elements of x which maximize [2.1.1] and satisfy [2.1.2]. Risk is introduced to the problem when either some elements of c, A or b or any combination of them include random elements.

Stochastic linear programming which was first suggested by Tintner (130) is primarily concerned with the statistical distribution of max $z = c'x$ in [2.1.1]. It is assumed that a multivariate probability distribution for the elements of A, b, c in [2.1.1] and [2.1.2] is known. We may be represented, for instance, as

$$\text{Prob } (A,b,c) \qquad [2.1.3]$$

where Prob refers to the probability of simultaneous occurrence of specified values of the matrix A and the vectors b and c.

Thus, given a Prob function as in [2.1.3] one may ask how max $c'x$ will be distributed. There are three types of distribution problems: the passive approach, active approach, termed distribution problems and expected values problems by

Vajda (136), and other distributional approaches.

The passive approach assumes that for each possible configuration of the random variables the optimal activities, x^0 are selected. Then it is possible, at least in principle, to derive the probability distribution

$$F(z) \tag{2.1.4}$$

of the linear form z to be optimized. To simplify the problem usually only the linear terms of the Taylor development of the function are retained. This is based on the assumption that the coefficients are independent. A confidence interval for the expected value of the function can be calculated (86, 84). This method has been developed by Babbar (6), Tintner (129) and extended by Sengupta (122, 109), Bereanu (13), Prekopa (99) and others (121). For example one way of obtaining a distribution of max c'x that has been used (27) is to use a Monte Carlo routine to generate the values of the components of b based upon the given statistical distribution of b. When a set of values for b is selected the values of x can be solved using regular linear programming methods.

The active approach consists of transforming the problem into a decision problem. The problem can be reformulated in the following manner

$$\text{Max } c'x$$

subject to

$$Ax \leq bU$$

$$x \geq 0$$

where U is matrix with elements such that

$$0 \leq \mu_{ij} \leq 1 \qquad i=1,2,\ldots m \qquad j=1,\ldots,n$$

and

$$\sum_{j=1}^{n} \mu_{ij} = 1$$

U is a matrix of decision variables, μ_{ij}, which denote the proportion of the resources i devoted to activity j. This also assumes that all resources are completely utilized.

In order to illustrate the method further we turn to the following example given in (88, 85) about an application to Iowa agriculture, the data are mentioned in detail by Babbar (6, 7). For other examples see (120, 108).

A farm that can either grow corn in amount x_1 or raise flax in amount x_2. The net price of corn is 1.56 and the net price of flax is 3.81. There are two production factors namely land 148 units and capital 1800 units, a_{11} and a_{12} indicate the amount of land used for producing one unit of x_1 and x_2 respectively. a_{21} and a_{22} indicate the amount of capital used for producing one unit of x_1 and x_2 respectively.

The passive problem would have us find the probability distribution of max z

$$z \equiv 1.56\ x_1 + 3.81\ x_2$$

subject to

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 148 \\ 1800 \end{bmatrix} \qquad [2.1.5]$$

when the probability distribution of

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{is specified.}$$

The active approach replaces the constraint [2.1.5] with

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \leq \begin{bmatrix} 148 & 0 \\ 0 & 1800 \end{bmatrix} \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix}$$

and then proceeds to examine the distribution of max z for various $\mu_{ij} \geq 0$ allocations which satisfy

$$\mu_{11} + \mu_{12} = 1$$

$$\mu_{21} + \mu_{22} = 1$$

The objective is to choose a best set of μ_{ij} values for the matrix u as judged relative to a suitable defined preference functional. The method has been applied in (25, 132).

The other distributional approaches (120) have been concerned with several specific aspects of the general problem of which the following may be mentioned.

(a) The extreme value aspects according to which each c_j coefficient of the objective function is replaced over the sample values (t=1,2,...,N) by $(\max_t c_{jt})$ or $(\min_t c_{jt})$ and the resulting distribution of the optimal solution and its effects on the optimal decision role are analyzed (106, 105).

(b) The implication of any departure from the assumption of normality, mutual independence and other simplifying conditions for the random elements of the problem; particularly the results which are relatively distribution-free are much needed (110).

(c) The distribution of basic feasible solutions other than the optimal one (111) and the distribution of the posterior distribution of optimal profit, when the prior probabilities on c_j coefficients are replaced by the decision-maker by posterior probabilities through sampling experiments with consequent change in the optimal solution (116).

(d) Distribution of the utility function u(z) of certain form, defined on set of values of the objective function z = c'x subject to the usual linear restrictions (18, 11).

Finally we review the earliest problem of the passive approach (114) in which the optimal basis equations are written as

$$(A+\alpha)x = (b+\beta) \qquad [2.1.6]$$

where A and α are mxm matrices and x, b and β are mx1 vectors and where the random errors α_{ij} and β_i are assumed to satisfy the following conditions

$$E(\alpha_{ij}) = E(\beta_i) = 0; \quad E\alpha_{ij}^2 = \sigma_{ij}^2 < \infty; \; E\beta_i^2 = t_i^2$$

From [2.1.6] the kth optimal activity x_k can be solved for as

$$x_k = \frac{|D^k+d^k|}{|A+\alpha|} \qquad k=1,\ldots m$$

where $|D^k+d^k|$ is the determinant of matrix $(A+\alpha)$ when its kth column is replaced by the column vector $(b+\beta)$.
We have also

$$z = \sum_{r=1}^{m} (c_r+\gamma_r)x_r$$

where

$$E\gamma_r = 0; \qquad E\gamma_r^2 = w_r^2 < \infty$$

and the index set r=1,2,...,m is defined for the optimal basic activities only.

If we are now limiting to first order errors and neglecting cross product terms of unlike random elements, the following expressions can be derived:

$$|D^k+d^k| \doteq |D^k| + \sum_{i=1}^{m} s_{ik}\beta_i + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq k}}^{m} \alpha_{ij}^k \alpha_{ij} = N(x_k)$$

$$|A+\alpha| \doteq |A| + \sum_i \sum_j s_{ij}\alpha_{ij} = D(x)$$

$$\sum_{r=1}^{m} (c_r+\gamma_r)|D^r+d^r| = \sum_r c_r|D^r| + \sum c_r(\Sigma s_{ir}\beta_i)$$

$$+ \sum_{\substack{j=1 \\ j\neq r}} c_r \sum d_{ij}^r\alpha_{ij} + \sum_r |D^r|\gamma_r = N(z)$$

and

$$E|D^k+d^k| = |D^k| = d_k$$

$$\mathrm{Var}|D^k+d^k| = \sum_i s_{ik}^2 t_i^2 + \sum_i \sum_j (d_{ij}^k)^2\sigma_{ij}^2 = \sigma_k^2$$

$$E|A+\alpha| = |A| = a_0$$

$$\mathrm{Var}|A+\alpha| = \sum_i \sum_j s_{ij}^2\sigma_{ij}^2 = \sigma_A^2$$

$$\mathrm{Cov}(|D^k+d^k|, |A+\alpha|) = \sum_i \sum_j d_{ij}^k s_{ij}\sigma_{ij}^2 = \sigma_{Ak}^2$$

$$E[N(z)] = \sum_r |D^r|c_r = d_{N(z)}$$

$$\mathrm{Var}\ N(z) = \sum_{r=1}^{m} (\sum_{i=1}^{m} c_i s_{ri})^2 t_r^2 + \sum_{r=1}^{m} C_r^2[\sum_{i=1}^{m} \sum_{j=1}^{m} (d_{ij}^r)^2\sigma_{ij}^2]$$

$$+ \sum_r |D^r|^2 w_r^2 = \sigma_{N(z)}^2$$

$z = N(z)|D(z)$ where $|A+\alpha| = D(x) = D(z)$.

with

$$E[D(z)] = |A| = a_0; \quad \text{Var } D(z) = \sigma_A^2 \text{ and}$$

$$\text{Cov}[N(z), D(z)] = \sum_r c_r \sum_{\substack{i,j \\ j \neq r}} d_{ij}^r s_{ij} \sigma_{ij}^2 = \sigma_{A.N(z)}^2$$

where s_{ik} is the cofactor of element a_{ik} in $|A|$ and d_{ij}^k is the cofactor of the element D_{ij} in $|D^k|$ and $|D^k|$ is the determinant of A with its kth column replaced by the resource vector b.

It is observed that since $N(x_k)$, $D(x_k)$, $N(z)$, $D(z)$ are linear functions of normally distributed variables, they themselves will be normally distributed, and the problem becomes one of finding the distribution of the ratios

$$x_k = \frac{N(x_k)}{D(x)} \text{ and } z = \frac{N(z)}{D(z)}$$

Using the result due to Geary (59), it is shown by Babbar and Tintner (7, 129) that the probability density of z for example is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} \frac{[\bar{D}\sigma_N^2 - \bar{N}\sigma_{\overline{ND}} + z(\bar{N}\sigma_D^2 - \bar{D}\sigma_{\overline{ND}})]}{[\sigma_D^2 z^2 - 2z\sigma_{ND} + \sigma_N^2]^{3/2}} \; .$$

$$\exp\left[-\frac{1}{2}\left\{\frac{(D_z - \bar{N})^2}{\sigma_D^2 z^2 - 2z\sigma_{ND} + \sigma_N^2}\right\}\right]$$

where $\bar{N}$ and D are two normally distributed variables with means $\bar{N}$, $\bar{D}$ and covariance σ_N^2, σ_D^2, $\sigma_{\overline{ND}}$.

Babbar (7) then discusses confidence intervals for the ratios, cumulative distributions, etc. and closes with a numerical application of these techniques to linear programming. Extension of this problem has been developed in (118, 119, 117).

2.2. Programming under Uncertainty

We have already pointed out the difference between uncertainty and risk since we are always going to use the name originally given to the method (36).

A class of linear programming models is considered where the activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that can be determined in advance because those in the second and later stages depend on the outcome of random events.

Madansky (76) has classified the methods of reducing the effects of uncertainty in a linear programming problem into three types.

(a) Replacing all random elements by their expected values the so-called expected value solution.

(b) Replacing the random elements by pessimistic estimates of their values in the second stage one could compensate for the loss due to inaccuracies in the first-stage decisions. The so-called fat solution.

(c) Reducing the problem to a two stage or multistage problem, where in the second stage one could compensate for the loss due to inaccuracies in the first-stage decisions. The so-called slack solution.

Finally we formulate the general multi-stage problem given in Dantzig (36). The structure assumed is

$$
\begin{aligned}
b_1 &= A_{11}x_1 \\
b_2 &= A_{21}x_1 + A_{22}x_2 \\
b_3 &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \\
b_4 &= A_{41}x_1 + A_{42}x_2 + A_{43}x_3 + A_{44}x_4 \\
&\;\;\ldots\ldots\ldots\ldots\ldots\ldots \\
b_m &= A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \ldots + A_{mm}x_m \\
c &= \phi(x_1, x_2, \ldots, x_m | E_2, E_3, \ldots E_m)
\end{aligned}
$$

where b_1 is a known vector; b_i is a chance vector $(i=2,\ldots,m)$ whose components are functions of a point E_i drawn from a multidimensional distribution; A_{ij} are known matrices. The sequence of decisions is as follows: x_1 the vector of non-negative activity levels in the first stage, is chosen so as to satisfy the first stage restrictions $b_1=A_{11}x_1$; the values of components of b_2 are chosen by chance by determining E_2; x_2 is chosen to satisfy the 2nd stage restrictions

$b_2 = A_{21}x_1 + A_{22}x_2$, etc. iteratively for the third and higher stages. It is further assumed that

(1) The components of x_j are non-negative;

(2) There exists at least one x_j satisfying the jth stage restraints, whatever be the choice of $x_1, x_2, \ldots, x_{j-1}$ satisfying the earlier restraints or the outcomes $b_1, b_2, \ldots, b_m$.

(3) The total cost **c** is a convex function in $x_1, x_2, \ldots x_m$ which depend on the value of the sample points $E_2, E_3, \ldots E_m$.

When the number of possibilities for the chance vector b_2 is $b_2^{(1)}, b_2^{(2)}, \ldots, b_2^{(k)}$ with probabilities $p_1, p_2, \ldots p_k$, $(\Sigma\, p_i=1)$, it is not difficult to obtain a direct linear programming solution for small k, say k=3. Since this type of structure is very special, it appears likely that techniques can be developed to handle large k. For k=3, the problem is equivalent to determining vectors x_1 and vectors $x_2^{(1)}, x_2^{(2)}, x_2^{(3)}$ such that

$$
\begin{aligned}
b_1 &= A_{11}x_1 \\
b_2^{(1)} &= A_{21}x_1 + A_{22}x_2^{(1)} \\
b_2^{(2)} &= A_{21}x_1 \qquad\qquad + A_{22}x_2^{(2)} \\
b_2^{(3)} &= A_{21}x_1 \qquad\qquad\qquad\qquad + A_{22}x_2^{(3)}
\end{aligned}
$$

$$\text{Min } E(c) = c_1x_1 + p_1c_2x_2^{(1)} + p_2c_2x_2^{(2)} + p_3c_3x_2^{(3)}$$

where for simplicity we have assumed a linear objective function.

Application of this method and extension may be found in (10, 75, 76, 40, 46, 136, 12, 137, 74).

2.3. Chance-Constrained Programming[1]

A new conceptual and analytical vehicle for problems of temporal planning under uncertainty, involving determination of optimal (sequential) stochastic decision rules is defined by Charnes and Cooper (22).

The problem of stochastic (or better, chance-constrained) programming is defined as follows. Select certain random variables with known distributions in such a manner as (a) to maximize a functional of both classes of random variables subject to (b) constraints on these variables which must be maintained at prescribed levels of probability. More loosely, the problem is to determine optimal stochastic decision rules under these circumstances. An example is supplied in (26). Temporal planning in which uncertainty elements are present, but in which management has access to "control variables" with which to influence outcomes, is a general way of characterizing these problems. Thus, queuing problems in which the

[1]This part is based on (100).

availability of servers, customers, or both are partly controllable fall within this classification. It should be noted that the constraints to be maintained at the specified levels of probability will typically be given in the form of inequalities.

Chance-constrained programming admits random data variations and permits constraint violations up to specified probability limits. Different kinds of decision rules and optimizing objectives may be used so that under certain conditions, a programming problem (not necessarily linear) can be achieved, that is deterministic in that all random elements have been eliminated. Existence of such "deterministic equivalent" in the form of specified convex programming problems is established for a general class of linear decision rules (24) under the following three classes of objectives: (1) maximum expected value ('E model'); (2) minimum variance ('V model') and (3) maximum probability ('P model').

A chance-constrained formulation would replace the ordinary linear programming problem with a problem of the following kind:

Optimize $f(c,x)=\text{Max } c'x$

Subject to $\text{Prob } (Ax \leq b) \geq \alpha, \quad x \geq 0$ [2.3.1]

A,b,c are not necessarily constant but have, in general, some or all of their elements as random variables. The vector α contains a prescribed set of constants that are probability measures of the extent to which constraint violations are admitted. Thus, an element $0 \leq \alpha_i \leq 1$ is associated with a constraint $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ to give

$$\text{Prob} \left(\sum_{j=1}^{n} a_{ij}x_j \leq b_i \right) \geq \alpha_i \qquad [2.3.2]$$

a double inequality which is interpreted to mean that the ith constraint may be violated but at most $\beta_i = 1-\alpha_i$ proportion of the time.

Here it is proposed to examine important classes of chance-constrained problems and to obtain deterministic equivalents that are then known in certain cases to be convex programming problems. It is to be emphasized, however, that optimization under risk immediately raises very important questions concerning a choice of rational objectives. Questions can arise, for example, concerning the reasonableness of an expected value optimization. Without attempting to resolve these issues, it should be noted that the evaluators secured for one objective are not necessarily correct or optimal when applied to the same problem under an altered objective.

It is assumed that a choice of values for decision variables x will not disturb the densities associated with the random variables in A,b,c. Then we may formulate the general problem in terms of choosing a suitable decision rule

$$x = \phi(A,b,c) \qquad [2.3.3]$$

with the function ϕ, to be chosen from a prescribed class of functions and applied in a manner that guarantees that x values, as generated, will satisfy the chance constraints of [2.3.1] and optimize f(c,x) in [2.3.1] with reference to the class of rules from which the ϕ of [2.3.2] is to be chosen.

By assuming that the matrix A is constant (i.e. non-random) x will also be restricted by the rule [2.3.3] to members of the class

$$x = Db \qquad [2.3.4]$$

where D is a n x m matrix whose elements are to be determined by reference to [2.3.1].

We will examine all possible rules of form D and, for important classes of objective and statistical distributions, in order to be able to characterize situations in which a deterministic equivalent will be achieved-irrespective of the D choice thus yielding a convex programming problem.

The expected value model ('E model') is then

maximize $E\ c'\ x$

under conditions $\text{Prob}\ (Ax \leq b) \geq \alpha$ [2.3.5] substituting [2.3.4] into the objective function of [2.3.5] one obtains

$$E(c'Db) = (Ec)'\ D(Eb)$$

It will assume that b and c are uncorrelated, then it will define the vectors

$$\mu_c' \equiv (Ec)' \ ; \quad \mu_b' \equiv (Eb)'$$

then

$$\text{Min } - \mu_c' D\mu_b$$

Denoting the ith row of the matrix A by a_i' and $(b-\mu_b)$ by $\hat{b}$ and assuming normality of distribution for the variates $(a_i', D\ \hat{b} - \hat{b}_i)$, parts of the constraints of [2.3.4] may be written as

$$\text{Prob } (a_i'Db - b_i \leq 0) = \text{Prob } (b_i - a_i'Db \geq 0)$$

$$= \text{Prob } (\hat{b}_i - a_i'D\hat{b} \geq \mu_{b_i} + a_i'D\mu_b) \geq \alpha_i$$

Assuming $E(\hat{b}_i - a_i'D\hat{b})^2 > 0$, the above can be normalized and ith constraint can be written fully as

$$\text{Prob } \frac{b_i - a_i'\ D\ \hat{b}_i}{\sqrt{E(\hat{b}_i - a_i\ D\ \hat{b})^2}} \geq \frac{-\mu b_i + a_i'\ D\ \mu_b}{\sqrt{E(\hat{b}_i - a_i\ D\ \hat{b})^2}} \geq \alpha_i \qquad [2.3.6]$$

by the assumption of normality, the left-hand side of the argument, i.e. $(\hat{b} - a_i{}'D\ \hat{b}_i) \ / \ \sqrt{E\ (\hat{b}_i - a_i{}'D\ \hat{b})^2}$ is a standardized normal variable with zero mean and unit variance, so that [2.3.6] is replaced by

$$F_i \quad \frac{-\mu_{b_i} + a_i{}'D\ \mu_b}{\sqrt{E(\hat{b}_i - a_i{}'\ D\ \hat{b})^2}} \qquad [2.3.7]$$

where

$$F_i(w) = (\sqrt{2\pi})^{-1} \int_w^\infty e^{-y^2/2}\, dy$$

Usually for normal distribution $\alpha_i \geq 0.5$ is taken, then the equation [2.3.7] can be solved as

$$\frac{-\mu_b + a_i{}'\ D\ \mu_b}{\sqrt{E(\hat{b}_i - a_i{}'\ D\ \hat{b})^2}} \leq F_i{}^{-1}(\alpha_i) \equiv -q_i \qquad [2.3.8]$$

where $q_i > 0$ for all i, if $\alpha_i > 0.5$.

The system [2.3.8] which involves nonrandom variables (i.e.), deterministic values only can be further reduced to a convex programming problem by introducing new variable v_i and writing [2.3.8] as

$$-\mu_{b_i} + a_i{}'D\mu_b \leq -v_i \leq -q_i \quad \sqrt{E(\hat{b}_i - a_i{}'D\ \hat{b})^2} \leq 0$$

or

$$\mu_{b_i} - a_i{}'D\mu_b \geq v_i \geq q_i \quad \sqrt{E(\hat{b}_i - a_i{}'D\ \hat{b})^2} \geq 0$$

which can be further simplified by squaring both sides, since nonnegativity is assigned to all expressions between inequality signs i.e.,

$$-a_i{}'D\mu_b - v_i \geq -\mu_{b_i}$$

$$-q_i{}^2\ E(\hat{b}_i - a_i{}'Db)^2 + v_i{}^2 \geq 0$$

with $v_i \geq 0$ for each i. Hence, the equivalent convex program for chance-constrained programming [2.3.5] is

Minimize $-\mu_c{}'\ D\mu_b$

under the conditions

$$\mu_{b_i} - a_i{}'D\ \mu_b - v_i \geq 0 \qquad [2.3.9]$$

$$-q_i{}^2\ E(a_i{}'Db - b_i)^2 + q_i{}^2(\mu_{b_i} - a_i{}'D\mu_b)^2 + v_i{}^2 \geq 0$$

where the problem [2.3.9] is a convex programming problem in the variables D and V

For the minimum variance ('V model')

Min $E(c'x - c^{0}{}'x^{0})^2$

under the conditions [2.3.10]

Prob $(Ax \leq b) > \alpha$

$x = Db$

where the objective is to minimize a generalized mean square error i.e., taking all relations between the c_j into account, it is intended to minimize this measure of their deviations about some given preferred values $z^0=c^{0}{}'x^0$.

It is easy to achieve the following deterministic equivalent to [2.3.10]

$$\text{Min} \quad E(c'Db-c\,'x')^2$$

under conditions

$$\mu_{b_i}-a_i{}'D\mu_b-v_i\geq 0$$

$$-q_i{}^2E(a_i{}'Db-b_i)^2+q_i{}^2(\mu_{b_i}-a_i{}'D\mu_b)^2+v_i{}^2\geq 0$$

$$v_i\geq 0$$

This deterministic equivalent is again a convex programming problem.

The maximum probability ('P model') turns to a version of the satisficing approach. In this approach the $c^{0}{}'x^0$ components are specified relative to some set of values - e.g., as generated from an aspiration level mechanism - which an organization (an individual or a business firm in the present context) will regard as satsifactory whenever these levels are achieved. Of course, when confronting an environment subject to risk, the organization cannot be sure of achieving these levels when effecting its choice from what it believes are available alternatives. On the other hand, if it does not achieve the indicated $c^{0}{}'x^0$ levels or, more precisely, if it believes that it cannot achieve them at a satisfactory level of probability, then the organization will either (a) reorient its activities and 'search' for a more favorable environment or else (b) alter its aspirations and, possibly, the probability of achieving them.

The model is Max Prob $(c'x \geq c^{0}{}'x^{0})$

under the conditions

$$\text{Prob}\ (Ax \leq b) \geq \alpha \qquad [2.3.11]$$

$$x = Db$$

If the same rules and assumptions are utilized as before to reduce this to a deterministic equivalent, it then becomes

Max v_0/w_0

under conditions

$$\mu_c{}'D\ \mu_b - v_0 \geq \mu_{c^0} \qquad [2.3.12]$$

$$-\ E(c'Db - c^{0}{}'x^{0})^2 + w_0{}^2 \geq 0$$

$$\mu_{b_i} - a_i{}'D\ \mu_b - v_i \geq 0$$

$$-q_i{}^2\ E(a_i{}'Db - b_i)^2 + q_i{}^2(\mu_{b_i} - a_i{}'D\ \mu_b)^2 - v_i{}^2 \geq 0$$

$$v_i \geq 0$$

This problem can be solved using fractional programming methods; for more details see (22).

Sengupta (55) points out two aspects which may be noted about this method. The first aspect is that it characterizes the problem only within a very restricted class of decision rules, and the operational efficiency of the method must be determined by further experimentation. In other words, one could specify other types of deterministic equivalents (11) which would subsume the cases considered here. Secondly, the decision rules here are not analytic, i.e., each time they have to be solved with the appearance of new data. An extension of this idea of deterministic equivalent in terms of recursive programming may be helpful, although it will involve nonlinear difference equations that are very difficult to solve.

Shinji Kataoka (71) introduced a new objective function, which is suitable for stochastic programming, utilizing Charnes' and Cooper's model. That is

$$\text{Max } f \qquad [2.3.13]$$

Subject to

$$\text{Prob } (c'x \leq f) = \alpha \qquad [2.3.14]$$

and

$$\text{Prob } (Ax \leq b) \geq \beta \qquad [2.3.15]$$

$$x \geq 0$$

It should be noted that the expected value of profit is not always considered a good measure for the optimality criterion. Even though a policy x dominates other policies in the expectation of profit, it may be more risky in that the chance of getting a very low profit may be greater than for other policies because of the dispersion of its distribution.

For this reason, the lower allowable limit f defined by [2.3.14] a special form of [2.3.15] for a given probability α is maximized instead of the expected value profit.

A case is considered in which the b_i's and c_j's are random variables, but the a_{ij}'s are constant. Transportation and production horizon problems belong to this category if customer demand and commodity price are random. This is called a transportation type problem.

Kataoka has made the following assumptions and formulations.

A.1. The random variable b_i has a normal distribution with mean value $\bar{b}_i$ and variance $\sigma_{b_i}^{\ 2}$

The probability in [2.3.15] can be transformed as

$$\text{Prob } (\sum_j a_{ij}x_j \leq b_i) = \text{Prob } (\frac{b-\bar{b}_i}{\sigma_{b_i}} \geq \frac{\sum_j a_{ij}x_j-\bar{b}_i}{\sigma_{b_i}})$$

then the left hand side of the argument is a normalized random variable with zero mean and unit variance. Hence the probability condition, [2.3.15] is replaced by

$$G(\frac{\sum_j a_{ij}x_j-\bar{b}_i}{\sigma_{b_i}}) \geq \beta_i$$

or

$$\sum_j a_{ij}x_j-\bar{b}_i \leq G^{-1}(\beta_i) = -q_i$$

where

$$G(x) = (\sqrt{2\Pi})^{-1} \int_x^\infty e^{-y^2/2}\, dy$$

usually it is considered that $\beta_i \geq 0.5$; then $G^{-1}(\beta_i) \leq 0$.

A.2. The vector c has a multinormal distribution with mean value vector $\bar{c}=(c_1, c_2 \ldots, c_n)$ and a dispersion matrix V. The variance of c'x is x'Vx. Hence

$$\text{Prob}(c'x \leq f) = \text{Prob}\left(\frac{c'x-\bar{c}x}{\sqrt{x'vx}} \leq \frac{f-\bar{c}'x}{\sqrt{x'vx}}\right) = I\left(\frac{f-\bar{c}'x}{\sqrt{x'vx}}\right)$$

where

$$I(x) = (\sqrt{2\Pi})^{-1} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

then for (2.3.14) is

$$f=\bar{c}'x+I^{-1}(\alpha)\sqrt{x'vx}$$

Finally Kataoka has a maximization problem

$$\text{Max } f_I=\bar{c}'x+I^{-1}(\alpha)\sqrt{x'vx} \qquad [2.3.16]$$

under the conditions

$$Ax \leq \bar{b}_i+G^{-1}(\beta_i)\sigma_{b_i}$$

Kataoka also transforms a model to a more general stochastic programming problem in which the components of matrix A are random variables; for more details see (71).

Sengupta (107) considers three generalized standpoints. First, the assumption of normality is replaced by a chi-square distribution, which has a nonnegative range and hence more applicability to economic problems of production planning; and a confidence interval for the optimal solution vector is worked out on this basis. Second, the relevance of chance-constrained programming to sensitivity analysis of optimizing economic models is briefly indicated. Third, the applicability of chance-constrained decision rules to problems of development planning through investment programming is discussed.

Sengupta (107) assumes that the elements a_{ij}, b_i, of A and b respectively are taken to be mutually independent chi-square variates with means $\bar{a}_{ij}$ and $\bar{b}_i$ and these are denoted by χ^2_{ij} (a_{ij}) and $\chi^2(b_i)$ respectively. He mentions two points about the reasonableness of this assumption. First, in most economic problems of production and resource allocation, the input coefficients a_{ij} represent coefficients of production function and therefore these must be nonnegative. Similarly, the resource vector must be nonnegative. Second, a chi-square, which is closely related to the normal (e.g., a normal variate truncated at $y \geq 0$ results in a chi-square) has properties very similar to a normal distribution (e.g., reproductive properties) and hence approximations can easily be worked out by means of normal tables whenever needed.

In the derivation of his model, Sengupta assumes for a moment that b is not random. By transformation [2.3.1] becomes

$$\text{Prob}(\chi_i^2(\sum_{j=1}^{n} \bar{a}_{ij}) \leq \frac{b_i \sum_j \bar{a}_{ij}x_j}{\sum_j \bar{a}_{ij}x_j^2}) \geq \alpha_i \qquad [2.3.17]$$

or, alternatively as,

$$F_i(b_i \sum_{j=1}^{n} \bar{a}_{ij}x_j / \sum_{j=1}^{n} \bar{a}_{ij}x_j^2) \geq \alpha_i \qquad (i=1,\ldots,n)$$

where f_i (w) is the cumulative distribution function of a central chi-square variate with degrees of freedom $N=\sum_j \bar{a}_{ij}$, i.e.

$$F_i(w)-(2^{n/2}\,\Gamma(N/2))^{-1} \int_0^w t^{(n/2)-1}.\ \exp(-t/2)\ dt$$

Since the ordinary chi-square tables give the various significance points for w for a given degree of freedom, it would be possible to compare the exact values of

$$W = b_i \sum_{j=1}^{n} \bar{a}_{ij}x_j / \sum_{j=1}^{n} \bar{a}_{ij}x_j^2$$

satisfying the inequality [2.3.17]. For example, if d_i=.990 (i.e., the tolerance measure) and $\sum_j \bar{a}_{ij}$=7.0, then from the chi-square table one finds that

$$\text{Prob}(\chi_i^2\ (7.0) \leq w_0) = .990$$

implies a value of w_0=18.4753. Therefore, if it is taken that $w \geq w_0$, this would satisfy a tolerance measure of 99% or higher. Since, for any preassigned value of tolerance measure α_i and the value of $N=\sum_j \bar{a}_{ij}$, one can find a positive value of w_0 from the chi-square table.

The chance-constrained programming model [].2.1] then is finalized as a convex programming problem of the following type.

$$\text{Minimize } -c'x = -\sum_{j=1}^{n} c_j x_j$$

under the conditions

$$b_i \sum_{j=1}^{n} \bar{a}_{ij} x_j - q_i \sum_{j=1}^{n} \bar{a}_{ij} x_j^2 \geq 0$$

where

$$q_i = w_0 \quad x_j \geq 0$$

For a general case, Sengupta uses the F distribution when b is also random and he obtains the following concave programming problem.

$$\text{Maximize } c'x = \sum_{j=1}^{n} c_j x_j$$

under the restrictions

$$\bar{b}_i \Big(\sum_{j=1}^{n} \bar{a}_{ij} x_j\Big) - k_i \Big(\sum_{j=1}^{n} \bar{a}_{ij} x_j^2\Big)\Big(\sum_{j=1}^{n} \bar{a}_{ij}\Big) \geq 0$$

$$x_j \geq 0$$

where K_i is obtained as follows.

$$r=\left(\sum_{j=1}^{n} \bar{a}_{ij}x_j^2\right)\left(\sum_j \bar{a}_{ij}\right)/\bar{b}_i\left(\sum_{j=1}^{n} a_{ij}x_j\right)$$

$$M_1=\sum_{j=1}^{n} \bar{a}_{ij};\ M_2=\bar{b}_i$$

therefore

$$\text{Prob}\ (F(M_1,M_2) \leq 1/r) \geq \alpha_i$$

then

$$K_1=1/r_0$$

Sengupta (107) considers that at the macroeconomic level, chance-constrained interpretations are most appropriate for the restrictions of a linear programming model applied to development planning. At the microeconomic level, the chance-constrained model is applicable most appropriately to situations of portfolio investment allocation and the holding of assets when a margin of safety is desired.

Further interesting results can be obtained assuming another kind of distribution with nonnegative range such as the exponential, the gamma or the beta distribution.

In the economic world disasters may occur. For a great many people, the idea of a disaster exists and the principle of "safety first" asserts that it is reasonable and probable in practice that an individual will seek to reduce as far as possible the chance of a catastrophe occurring.

A single disaster is a discontinuity in one's pattern of behavior and in one's scale of preferences, viz. death, bankruptcy or a prison sentence.

A. D. Roy (102) has developed the safety first principle in terms of minimizing the upper bound of the chance of a dread event, where the information available about the joint probability distribution of future occurrences in confined to the first and the second order moments only.

From a formal standpoint, the minimization of the chance of a disaster can be interpreted as maximizing expected utility if the utility function assumes only two values, e.g. one if disaster does not occur, and zero if it does. It would appear that this formal analogy is scarcely helpful, since in the one case an individual is trying to make the expected proportion of occurrences of disaster as small as possible, while in maximizing expected utility he is operating at a different level of satisfaction.

A complete hypothesis about individual or corporate economic behavior under uncertainty must specify three things. It must describe the way in which expectations are formed from experience of the hard facts of life, the objectives which the entity under examination is trying to achieve, and the opportunities present for attaining such ends.

It may be possible that the outcome of economic activity which is regarded as disaster, is not independent of the expected value of the outcome. Thus, a person may be prepared to revise the level of disaster downwards if the expected return is at the same time raised. For example, he may at one and the same time regard a speculative loss of 10 percent as a disaster if the expected gain is only 5 percent, while, if the expected gain is 15 per cent, he will only get excited if his loss exceeds 25 per cent. Once again, such individual psychology can no doubt be interpreted in terms of utility function, but such development will not be pursued here. In the following discussion, the disaster level of the outcome is taken to be constant.

Let it be supposed, then, that the principle of safety first is adopted and that, when confronted with a range of possible actions, we are concerned that our gross return m should not be less than some quantity d. With every possible action, this outcome is not certain. There is coupled with m a quantity σ (the standard error of m) which is, very roughly, the average amount by which the prediction m is expected to be wrong. In the following, it is assumed that m and σ are known precisely, whereas in fact they must be estimated from information about the past. This raises all kinds of problems, which are beyond the scope of this discussion, since estimates of m and σ, say $\hat{m}$ and $\hat{\sigma}$, will themselves have sampling distributions.

Thus a full analysis of the problem should discuss simultaneously not only behavior under uncertainty but also actions under uncertain uncertainty.

In the particular application of the principle of safety first which is examined here, (102), it is postulated that m and σ are the quantities that can be distilled out of our knowledge of the past. The slightest acquaintance with problems of analyzing economic time series will suggest that this assumption is optimistic rather than unnecessarily restrictive.

Given the values of m and σ for all feasible choices of action, there will exist a functional relationship between these quantities, which will be denoted by $F(\sigma,m)=0$. There may be a whole family of such relationships; in this case $F(\sigma,m)=0$ is their envelope. Since it is not possible to determine with this information the precise probability of the final return being d or less for a given pair of values of m and σ, the only alternative open is a calculation of the upper bound of this probability. This can be done by an appeal to the Bienaymé-Tchebycheff inequality. Thus, if the final return is a random variable z then

$$\text{Prob}\ (|z-m| \geq m-d) \leq \frac{\sigma^2}{(m-d)^2}$$

If, then, in default of minimizing $P\ (z \leq d)$, one operates on $\sigma^2/(m-d)^2$, this is equivalent to maximizing $(m-d)/\sigma$.

Telser (128) postulates a particular attitude toward risk with stems from Roy's paper dealing with the theory of asset holding. He asks what assumptions make about entrepreneurial behavior in the face of uncertainty and whether or not entrepreneurs maximize their expected income. Suppose an entrepreneur wishes to select a portfolio of assets so as to maximize expected net income. Then he would buy only one asset, namely, that whose price is expected to increase the most. If he is right, he would gain a great deal, but conversely, if he is wrong he would lose a great deal. It has been observed that people diversify their portfolios, hence reject the hypothesis that entrepreneurs maximize expected net income.

However, entrepreneurs do prefer larger net incomes to smaller net incomes. Suppose an entrepreneur considers all his actions and strategies and for each action calculates the probability that the income resulting from the action, which is a random variable, falls short of a disaster level. For each action a there is a probability distribution of net income I which can be written Prob $(I \leq c;\ a)$ where c is some constant. One computes the Prob $(I \leq r;\ a)$ where $o \leq p \leq 1$, and r is the disaster level of income. This disaster level of income, r, could be associated with bankruptcy or with something less dramatic.

Suppose that the entrepreneur does not want the probability of his net income falling short of r to exceed α. Hence he will not choose any action such that Prob $(I \leq r;\ a) = p > \alpha$. By this means, all his actions can be put into one of two classes. The first class consists of all the actions a such that Prob $(I \leq r;\ a) > \alpha$ and the second class consists of all the actions a such that the Prob $(I \leq r;\ a) \leq \alpha$. All the actions in the second class shall be called admissible.

Then the entrepreneur will choose that action a of the admissible actions such that his expected income is at a maximum. Mathematically this means that the entrepreneur chooses the action a so that:

$$\max_{a} \Sigma I$$

Subject to

$$\text{Prob } (I \leq r;\ a) \leq \alpha$$

It would appear that such a rule of behavior requires that the entrepreneur knows the probability distribution of I for any action a that he might choose.

Fortunately we may appeal to the Tchebycheff inequality which permits one to set an upper bound to the Prob $(I \leq r;\ a)$ even when one does not know the probability distribution of I. The Tchebycheff inequality permits one to assert that:

$$\text{Prob } (|I-\bar{I}| \geq K) \leq \frac{\sigma^2}{K^2}$$

where $K>0$, σ^2=variance of I and $\bar{I}$=mean of I

It is not hard to show that

$$\text{Prob } (I \leq r) \leq \frac{\sigma^2}{(\bar{I}-r)^2}$$

This means that when $\frac{\sigma^2}{(\bar{I}-r)^2} \leq \alpha$ then Prob $(I \leq r) < \alpha$
Accordingly,

$$\frac{\sigma^2}{(\bar{I}-r)^2} \leq \alpha$$

becomes the risk restriction which is used.

It is assumed that the entrepreneur knows σ^2 and $\bar{I}$ for each a, but that he knows nothing more about the probability distribution of I for each a.

This formulation of the safety - first principle differs from that of A. D. Roy. He assumes that entrepreneurs minimize the probability of disaster. If they did, then their expected net income for that action which minimized the probability of disaster could be less than zero, i.e. they could be expected to lose money on their portfolio. This implies that there is no asset which the entrepreneur can hold without risk, that is, without the chance of gain or loss.

Sengupta (110) attempts to generalize the decision rules under chance-constrained programming from the viewpoint of safety first principles based on Tchebycheff-type probabilistic inequalities. The latter inequalities are utilized to define distribution free tolerance levels. The optimization criterion of chance-constrained programming based on the mean and variance is extended to a more generalized formulation based on the Kolmogorov-Smirnov's statistic on the maximum discrepancy of the population and sampling distributions.

Application of chance constrained programming can be found in (30, 29, 83, 89, 64, 1, 32, 31, 28).

2.4. Reliability Programming[1]

Sengupta (113) has extended the chance constrained programming method, interpreting each chance-constraint as a reliability measure. In this approach a tolerance level (in terms of a probability measure), one for each probability constraint is preassigned by the decision-maker and this set of tolerance measures is supposed to indicate the limit up to which constraint violations are permitted. This view of probabilistic linear constraints allows the interpretation of a linear programming model as a system, where each probabilistic constraint can be interpreted as a system component, where each component may have different degrees of reliability (e.g., different tolerance measures). The system reliability would then be dependent on the reliability of its individual components, their interdependence and the choice of the decision variables. Sengupta's objective is to charac-

[1]This section is based on Sengupta (113, 112, 115).

terize in a simplified framework the method of attaining and optimizing the reliability of a linear programming system under alternative probabilistic variations of constraint parameters.

Sengupta's system reliability approach has two basic differences from the chance-constrained approach. First the system reliability is not preassigned but has to be solved for in an optimal fashion. Second, the statistical theory of system reliability (73, 142), which is essentially based on non-negative life distribution is developed (113) in the production and resource allocation framework of economy theory.

He first considered a linear programming system where each linear constraint is probabilistic only in its resource availabilities. The L.P. system is

$$\max_{a} z = c'x$$

where

$$x \varepsilon X \equiv \{x \mid Ax \leq b,\ x \geq 0\} \qquad [2.4.1]$$

The resource vector $b=(b_i)$ is assumed random with probabilistics for its components satisfying the following assumptions:

<u>Assumption 1</u>: The probability distribution of each b_i is continuous with a non-negative range and mutually independent such that the cumulative distribution

$$F(a_i'x) = \text{Prob}(b_i \leq a_i'x) = 1-\mu_i; \quad 0 \leq \mu_i \leq 1 \qquad [2.4.2]$$

where a_i' denotes the ith row of matrix A in [2.4.1], is assumed known or estimable.

Assumption 2: The non-negative range of the distribution of each b_i is taken to be $[0,\infty]$ such that $R_i(0)=1$, $R_i(\infty)=0$ where

$$R_i(t) = 1-F(t) = \text{Prob }(b_i \geq t) = \mu_i; \quad t=a_i'x \qquad [2.4.3]$$

It is also assumed that the derivate of $R_i(t)$ exists for each i such that $r_i(t)dt$ where

$$r_i(t) = \frac{d\ R_i(t)}{dt} \qquad [2.4.4]$$

can be interpreted as the probability that the ith component (i.e. constraint) will fail (i.e. will be violated) at some time in the interval (t,t+dt). Note that $R_i(t)$ is a monotone non-increasing function in the interval $[0,\infty]$ and it can be interpreted as a measure of reliability.

By assumption one and two the system reliability R for the linear programming problem [2.4.1] with b random can be written in one possible formulation as

$$R_0 = \prod_{i=1}^{m} R_i(t) = \prod_{i=1}^{m} \mu_i; \quad 0 \leq \mu_i \leq 1 \qquad [2.4.5]$$

Sengupta pointed out that this is one of several possible forms of specification of the system reliability for a linear programming system which is based on three main assumptions.

(i) The linear programming system consisting of one chance constraints is assumed reliable, if and only if each ith constraint is feasible and reliable up to the tolerance level μ_i (i=1,...,m)

(ii) The system reliability R_0 in (2.4.5) is bounded in the interval [0,1], as also the components reliabilities R_i or μ_i; hence

$$1 \geq \max_i R_i \geq R_0$$

$$0 \leq \min_{i \atop m} R_i \geq R_0$$

when $R_0 = \prod_{i=1}^{m} \mu_i$

This implies that the system reliability cannot have a value higher than that of its most reliable component, in other words any component having zero reliability automatically makes the system reliability take on a zero value.

(iii) The system reliability measure R is based on the probability of simultaneous non-violation of all m chance-constraints, each with its reliability level μ_i (i=1,...,m). A measure R^0 based on the probability of simultaneous violation of all m chance-constraints would be

$$R^0 = 1 - \prod_{i=1}^{m} (1-R_i(t)) = 1 - \prod_{i=1}^{m} (1-\mu_i)$$

If we restricted for the present to a measure R of system reliability defined in [2.4.5] and [2.4.3], and the associated monotonic pay-off function for achieving the level R of system reliability and then we may hypothesize as in the portfolio analysis model (79, 80) and others (45) a utility function $u=u(R,z)$ of the decision maker with reliability (R) and profits ($z=c'x$) as two arguments satisfying the usual conditions

$$\frac{\partial u}{\partial z} > 0, \quad \frac{\partial u}{\partial R} > 0 \qquad \frac{dR}{dz} < 0$$

which imply risk aversion and guarantee in cases the u function is concave in its arguments the attainment of its maximum value.

A transformed programming problem which incorporates this system reliability may be presented as follows

$$\text{Max } u = w_, c'x + w_2 \sum_{i=1}^{m} \ln \mu_i \qquad [2.4.6]$$

subject to

$$R_i(a_i'x) \geq \mu_i \qquad [2.4.7]$$

$$0 \leq \mu_i \leq 1$$

$$x \geq 0$$

for $i=1,\ldots,m$

For instance if each b_i followed a two-parameter exponential density

$$p(b_i) = \lambda_i e^{-\lambda(b_i-\theta_i)} ; \qquad b_i \geq \theta_i \geq 0$$

with parameters $\lambda_i>0$ and $\theta_i>0$, then the reliability constraint (2.4.7) becomes

$$e^{-\lambda(a_i'x-\theta_i)} \geq \mu_i$$

taking logarithms we have

$$-\lambda(a_i'x-\theta_i) \geq \ln \mu_i$$

or

$$a_i'x \leq \theta_i - \frac{1}{\lambda_i} \ln \mu_i$$

On defining $x_{n+i}=-\ln \mu_i$, the transformed programming problem will be

$$\text{Max } u = w_1 \sum_{j=1}^{n} c_j x_j - w_2 \sum_{i=1}^{m} x_{n+i}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - \frac{1}{\lambda_i} x_{n+i} \leq \theta_i \; ; \quad i=1,2,\ldots,m.$$

$$x_j \geq 0 \qquad j=1,\ldots,n+m$$

it is possible to add lower bounds to x_{n+i} variables or any other class of restriction to them.

It is possible to consider several possible distribution forms satisfying the above assumptions. Using distributions other than exponential will produce nonlinear programming problems.

3. OPTIMIZATION TECHNIQUES

Nonlinearities frequently arise when a programming model's coefficients are viewed as random variables. For instance we consider any linear programming model in which the b_i coefficients in the linear inequality constraints are random. To illustrate, suppose the model has two constraints and that the coefficients b_i are independently distributed, where $G_i(b)$ represents the probability that the random variable b_i is at least as large as b. Suppose we want to select the x_j so that the joint probability of every constraint being satisfied is at least β:

$$\mathrm{Prob}\left[\sum_{j=1}^{n} a_{ij}x_j \leq b_1\right] \cdot \mathrm{Prob}\left[\sum_{j=1}^{n} a_{2j}x_j \leq b_2\right] \geq \beta \qquad (0<\beta\leq 1)$$

The programming constraints equivalent to the above one can be written as

$$\sum_{j=1}^{n} a_{ij}x_j - y_i = 0 \qquad \text{for } i=1,2$$

$$G_1(y) \cdot G_2(y) \geq \beta$$

where the last product leads to a nonlinear restriction on y_1 and y_2.

The nonlinear programming problem is to determine a vector x^0 that solves the problem

$$\text{Minimize } f(x) \tag{3.0.1}$$

subject to

$$g_i(x) \geq 0 \quad i=1,2,\ldots,m$$
$$j=1,2,\ldots,p$$

where any of the functions $f(x)$ and $\{g_i\}$ may be nonlinear.

In this chapter we will only indicate three techniques used in this thesis and they are sequential unconstrained minimization techniques, geometric programming and generalized polynomial programming.

3.1. Sequential Unconstrained Minimization Techniques

Anthony V. Fiacco and Garth P. McCormick (49) proposed to transform a mathematical programming problem [3.0.1] into a sequence of unconstrained minimization problems based on an idea proposed by C. W. Carroll (21) as follows:

Define the function

$$P(x,r) = f(x) + r_1 \sum_{i=1}^{m} \frac{1}{g_i(x)} \tag{3.1.1}$$

where r_1 is a positive constant.

Consider the following condition imposed on the programming problem.

C1: $R^0 = \{x | g_i(x) > 0, \quad i=1,2,\ldots,m$ is nonempty

C2: The functions $f_1, g_1, \ldots, g_m$ are twice continuously differentiable

C3: For every finite $k, \{x | f(x) \leq k;\ x \varepsilon R\}$ is bounded set, where $R = \{x | g_i(x) \geq 0,\ i=1,\ldots,m\}$.

NOTE: Conditions C2-C3 imply the existence of a finite number v_0 where

$$v_0 = \inf_{x \varepsilon R} f(x) = \operatorname*{Min}_{x \varepsilon R} f(x)$$

C4: The function f is convex

C5: The functions $g_1, \ldots g_m$ are concave

C6: The function $P(x,r) = f(x) + r \sum_i \frac{1}{g_i(x)}$ is, for each $r>0$, strictly convex for $x \varepsilon R^0$

Conditions C4 and C5 imply the convexity of P in R^0. The strict convexity is satisfied if:

(1) f is strictly convex; or

(2) any g_i is strictly concave; or

(3) there exist n independent linear constraints to the problem; or

(4) the requirement (a special case of (3)) that $x_i \geq 0$ for $i=1,\ldots,n$ are included in the problem.

For every convex programming problem there is an associated problem called its dual.

Primal problem

$$\text{Min } f(x) \qquad [3.1.2]$$

subject to

$g_i(x) \geq 0, \quad i=1,\ldots,m$.

The formulation of the dual is due to Wolfe (280, 281)

$$\text{Max } G(x,u)=f(x) - \sum_i u_i g_i(x) \qquad [3.1.3]$$

subject to

$$\nabla_x G(x,u)=\vec{0}, \qquad [3.1.4]$$

$$u_i \geq 0 \quad i=1,\ldots,m$$

If the primal problem has a solution at a point $\bar{x}$, then there exists $\bar{u}$ such that $(\bar{x},\bar{u})$ is a solution to the dual, and the extreme values of the problems are equal. In order to prove this it must impose requirements C4 and C5 on the programming problem in addition to differentiability and the Constraint Qualification of Kuhn and Tucker (72).

Theorem 1: (Fiacco-McCormick)

If

(1) R^0 is non-empty

(2) $f(x)$ and $-g_i(x)$, $i=1,\ldots,m$ are convex and twice continuously differentiable

(3) for every finite k $\{x|f(x)\leq k;\ x\varepsilon R\}$ is a bounded set

(4) for every $r>0$, $P(x,r)$ is strictly convex, then

(a) each function $P(x,r_k)$ for $r_k>0$ is minimized over R^0 at a unique $x(r_k)\varepsilon R^0$ where

$$\nabla_x P[x(r_k),r_k]=\vec{0} \qquad [3.1.5]$$

(b) $\lim_{k\to\infty} f[x(r_k)]=v_0$

where

$$v_0 = \inf_{x\varepsilon R} f(x) = \min_{x\varepsilon R} f(x)$$

As a consequence of equation [3.1.5] the method yields a dual-feasible point at each P minimum, where $u_i(r_k)=r_k|g_i^2[x(r_k)]$; $i=1,\ldots,m$.

Theorem 2: (Fiacco-McCormick)

Under the conditions of Theorem 1 the method yields dual feasible points $[x(r_k),u(r_k)]$ and $\lim_{k\to\infty} G[x(r_k), u(r_k)]=v_0$

Since v_0 is the maximum value of $G(x,u)$ for dual feasible points, the following inequalities obtain

$$G[x(r_k),u(r_k)]\leq v_0\leq f[x(r_k)]$$

These bounds are of considerable practical importance in deciding when to terminate convergence.

Fiacco and McCormick have further developed this approach which is sometime referred to by the acronym SUMT. For more information the reader can see (48, 51, 47, 81, 141, 78).

3.2. Geometric Programming

Geometric programming is a new mathematical programming technique which provides a systematic method for formulating a class of nonlinear optimization problem. The class of functions which geometric programming deals with are positive polynomials or posynomials for short.

Geometric programming is a technique which finds the optimum value without knowing the corresponding policy variables: instead of seeking the optimal values of the independent variables first, it finds the optimal way to distribute the total cost among the various terms of the objective function. Once the optimal allocations are defined, often by inspection of simple linear equations, the optimal cost can be found by routine calculation.

Richard Duffin, Elmor Peterson and Clarence Zener (39) are the authors of geometric programming, one of the most refreshing developments in optimization theory since the invention of the calculus. Geometric programming derives its name from its intimate connection with geometrical concepts, the most important being orthogonality of vectors.

To treat the problem of minimizing a posynomial, we employ the inequality which states that the arithmetic mean is at least as great as the geometric mean. We have that

$$(x_1-x_2)^2 \geq 0$$

$$x_1^2-2x_1x_2+x_2^2 \geq 0$$

adding $4x_1x_2$ to both sides

$$x_1^2+2x_1x_2+x_2^2 \geq 4x_1x_2$$

$$(x_1+x_2)^2 \geq 4x_1x_2$$

taking square root

$$x_1+x_2 \geq 2\sqrt{x_1x_2}$$

$$\frac{x_1+x_2}{2} \geq \sqrt{x_1x_2}$$

The Geometric Programming problem is stated as (38):

Primal program:

Find a minimum value of a primal function $g_0(t)$ subject to the natural constraints

$$t_1>0,\ t_2>0,\dots,t_m>0$$

and forced constraints

$$g_1(t) \leq 1,\quad g_2(t) \leq 1,\dots,g_p(t) \leq 1 .$$

where

$$g_k = \sum_{i \varepsilon J[k]} c_i t_1^{a_{i1}} t_2^{a_{i2}} \dots . t_m^{a_{im}} \qquad k=0,1,\dots,p$$

and

$$J[k]=\{m_k, m_{k+1}, m_{k+2}, \ldots, n_k\} \qquad k=0,1,\ldots,p$$

where

$$m_0=1, \quad m_1=n_0+1, \quad m_2=n_1+1, \ldots, m_p=n_{p-1}+1, \quad n_p=n$$

The exponents a_{ij} are arbitrary real numbers, but the coefficients c_i are assumed to be positive. Thus the function $g_k(t)$ are posynomials.

The dual program corresponding to the primal program is stated as:

<u>Dual program</u>:

Find the maximum value of a product function

$$v(\underline{\delta}) = \left[\prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right] \prod_{k=1}^{p} \lambda_k(\underline{\delta})^{\lambda_k(\underline{\delta})}$$

where

$$\lambda_k(\underline{\delta}) = \sum_{i \varepsilon J[k]} \delta_i \qquad k=1,2,\ldots,p$$

and

$$J[k] = \{m_k, m_{k+1}, m_{k+2}, \ldots, n_k\{, \qquad k=0,1,2,\ldots,p$$

where

$$m_0=1, \; m_1=n_0+1, \; m_2=n_1+1, \ldots, m_p=n_{p-1}+1, \; n_p=n$$

The factor c_i are assumed to be positive and the vector variable $\delta=(\delta_1,\ldots,\delta_n)$ is subject to linear constraints:

positive condition

$$\delta_1 \geq 0,\ \delta_2 \geq 0, \ldots, \delta_n \geq 0$$

normality condition

$$\sum_{i \varepsilon J[0]} \delta_i = 1$$

and orthogonality condition

$$\sum_{i=j}^{n} a_{ij}\delta_i = 0 \quad j=1,2,\ldots,m$$

the coefficients a_{ij} are real numbers.

The difference between the number of variables and the number of independent linear equations is conventionally called the number of degrees of freedom, as we have seen there are n orthogonality conditions, one for each variable t_i, a single normality condition and m weights, one for each term. Hence the equations have m-(n+1) degrees of freedom. Duffin and Zener (39) suggest calling this quantity the degree of difficulty.

Wilde (138) says that geometric programming is a challenging situation for any reader who has gotten this far into the subject, for most systems can be decomposed into components, each with its own cost or revenue. Component behavior in most engineering systems can usually be expressed as products of powers of the design variables, as it is testified by the frequency of logarithmic graphs in the technical literature. The effect of scale of operation is often expressed by such approximations as the six-tenths rules (19), which states that the cost of a piece of equipment varies as the $\frac{6}{10}$ power of its capacity. For problems with few degrees of difficulty, geometric programming promises to yield fast, accurate solutions to horribly nonlinear problems. The reader should refer to (38, 37, 19, 143, 144, 5) for further information.

3.3. Generalized Polynomial Programming

The positive coefficients of the posynomials are needed because they are raised to fractional powers in the geometric inequality, an operation forbidden to negative numbers. Passy and Wilde (98) developed the quasiduality theory of generalized polynomial programming. Geometric programming is now applicable to any problem involving generalized polynomials (negative coefficients permitted) in the objective function. Generalized polynomial inequalities of either sense can also be handled.

Passy and Wilde considered M+1 real generalized polynomial functions $g_m(x)$ of N real positive primal variables x_n:

$$g_n(x) \equiv \sum_{t=1}^{T_m} \sigma_{mt} c_{mt} \prod_{n=1}^{N} x_n^{a_{mtn}} \qquad m=0,1,\ldots,m$$

where

$$\sigma_{mt} = \pm 1$$
$$c_{mt} > 0$$
$$x_m > 0$$

and the a_{mtn} are any real numbers. Assume the signum function σ_{mt}, the coefficients c_{mt}, t_m (the number of terms in g_m), and the a_{mtn} are all given. Then the optimization problem will be

$$\min_{x>0} g_0(x) \equiv g_0(x^0) \equiv g_0^0$$

subject to

$$0<\sigma_m g_m^{\sigma_m} \leq 1 \qquad m=1,\ldots,M .$$

Where the σ_m are known signum functions. The functions g_m are assumed to satisfy the Kuhn-Tucker constraint qualification (72). Finding x^0 is in general a nonconvex programming problem for which no good general solution method exists.

They show that in nontrivial circumstance this problem can be solved by working with a set of real finite auxiliary variables δ_{mt}, one for each term of the g_m, which must all be finite and nonnegative for $m=0,1,\ldots,M$

$$0\leq\delta_{mt}<\infty;$$

and must satisfy the linear normality condition:

$$\delta_{00} \equiv \sigma_0 \sum_{t=1}^{T_0} \sigma_{0t}\delta_{0t} = 1; \qquad [3.3.1]$$

and the N linear orthogonality conditions:

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} a_{mtn} \delta_{mt} = 0 \qquad n=1,\ldots,N; \qquad [3.3.2]$$

and the M linear inequality constraints

$$\delta_{m0} \equiv \sigma_m \sum_{t=1}^{T_m} \sigma_{mt}\delta_{mt} \quad 0 \qquad m=1,\ldots,M;$$

with the qualification that

$$\delta_{mt} = 0$$

if and only if

$$\delta_{m0} = 0$$

where σ_0 is not specified in advance and must be chosen to satisfy the constraints.

From these auxiliary variables is formed the product function

$$v(\underline{\delta}, {}_0) \equiv \sigma_0 [\prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{c_{mt}\delta_{mt}}{\delta_{mt}}\right)^{\sigma_0}$$

where $\underline{\delta}$ gathered the finite auxiliary variables δ_{mt}.

They show that if $g_0(x)$ has a minimum at a finite point in T, the domain of definition of $\underline{x}$ then there exists auxiliary variables δ^0, σ^0 such that

$$v(\delta^0, \sigma_0^0) = g_0^0$$

and the optimal x^0 can be found by solving any N independent equations chosen from among the following equations, linear in the variables log x_n:

$$\sum_{n=1}^{N} a_{0tn}(\log x_n) = \log\left(\frac{\sigma_0 \delta_{0t} g_0^0}{c_{0t}}\right) \qquad t=1,\ldots,T_0$$

$$\sum a_{mtn}(\log x_n) = \log\left(\frac{\delta_{mt}}{\delta_{m0} c_{0t}}\right) \qquad t=1,\ldots,T_m$$

$$m=1,\ldots,M, \ \delta_{mo} \neq 0$$

From these logarithms the x^0 are readily found, when

$$\sum_{m=0}^{N} T_m = N+1$$

The system of linear equations [3.3.1], [3.3.2] has a unique solution (degree of difficulty equal to zero). As the number of degree of difficulty

$$D = \sum_{m=0}^{M} T_m - N - 1$$

increases, nonlinear computations and some trial and error methods are needed to find δ^0 and σ_0^0.

A computer program of generalized polynomial programming is available in Blau (16). For further information the reader can see (97, 16, 98).

4. ECONOMIC APPLICATIONS

4.1. Production Planning

Production planning (58) is concerned with specifying how the production resources of the firm are to be employed over some future time period in response to the predicted or forecasted demand for the product or services. The objective of production planning is to minimize the total cost of meeting demand within the constraints of a given system design. It could, of course consider the question of re-designing the system or adding physical capacity as part of the planning problem but these issues are here arbitrarily eliminated from consideration.

C. C. Holt, F. Modigliani J. F. Muth and H. A. Simon (68, 67) at the Carnegie Institute of Technology in 1955, conducted perhaps the most comprehensive experiment in production planning. The problem was to determine a linear decision rule for making production and labor-force decisions in successive time periods. The planning variables are capacity factors which can be manipulated by management and for which capital investment is not important, that would minimize the expected value of total cost over a large number of periods.

The balancing labor cost of changing labor force and inventory connected cost are assumed to be quadratic functions of the production quantities P_t and work force levels W_t. The following notation is used

W_t = the number of workers required for the period t; $t=1,2,\ldots,T$

P_t = the number of units to be produced for the period t

I_t = the inventory minus backlogs at the end of period t

S_t = forecast of the number of units to be ordered for shipment during the period t

c_k = cost coefficients to be determined for a given plant or situation (constants)

Specifically, the cost elements for period t are

Regular payroll	$[c_1W_t$
Hiring and layoff	$+ c_2(W_t-W_{t-1})^2$
Overtime	$+ c_3(P_t-c_4W_t)^2+c_5P_t-c_6W_t$
Inventory and shortage	$+ c_7(I_t-c_8-c_9S_t)^2]$

when $t=1,2,\ldots,T$ and the inventory level I_t obeys the

rule $I_t=I_{t-1}+P_t-S_t$

After the total cost function is obtained, the decision rule that minimizes the expected value of the total cost function is obtained by differentiating with respect to each decision variable. This result is a set of linear equations whose solution, in turn yields the decision rule. It is noted that the assumption of quadratic cost function results in linear equations after differentiating, and finally in linear decision rules. The obvious advantage is that differential calculus may be employed to arrive at a system of linear equations for optimal quantities P_t and W_t.

The resulting decision rules are of the following form.

$$P_t = \sum_{t=1}^{T} a_t S_t + bW_{t-1} + c\, I_{t-1} + d$$

$$W_t = \sum_{t=1}^{T} l_t S_t + f_t W_{t-1} + g_t I_{t-1} + h$$

Where all coefficients only depend on the cost parameters and may therefore be determined once for all as long as the cost parameters do not change.

An implicit assumption of decision analysis is that the underlying cost structure remains constant over many periods. It is not always realistic to make an estimate of the cost structure once and for all, the quadratic cost estimate will, from time to time, require revision in order to be reasonably accurate (67, p. 90). Other assumptions are perfect competition, homogeneous work and one product firm.

The cost coefficients used below will be the same as used by Holt, Modigliani, Muth and Simon in the paint factory (67, p. 73) and they are

$c_1 = 340$

$c_{13} = 0$

$c_2 = 64.3$

$c_{11} = 0.$

$c_3 = 0.20$

$c_4 = 5.67$

$c_5 = 51.2$

$c_6 = 281.$

$c_{12} = 0.$

$c_7 = 0.0825$

$c_8 = 320.$

$c_9 = 0.$

4.1.1. A reliability model

A system reliability approach to the following production programming model is developed here for the case of chance-constrained restrictions.

The problem is:

$$\text{Min} \sum_{t=1}^{T} [(c_1-c_6)W_t+c_2(W_t-W_{t-1})^2+c_3(P_t-c_4W_t)^2$$

$$+c_5P_t+c_7(I_t-c_8-c_9\bar{S}_t)^2]$$

subject to

$$\text{Prob}(P_t+I_{t-1} \geq S_t) \geq u_t \qquad t=1,\ldots,T$$

$$0 \leq u_t \leq 1$$

where the variable sales S is a random variable assumed to be a two-parameter exponential density function

$$f(S) \equiv f(S,\mu,\theta) = \mu e^{-\mu(S-S_0)} \qquad S_0 \leq S \leq \infty$$

where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \theta = S_0$

$$\theta = \text{minimum sale} = S_0$$

and μ is evaluated as:

$$E(S) = \int_{S_0}^{\infty} S\ \mu e^{-\mu(S-S_0)}\ dS = \mu e^{\mu S_0} \int_{S_0}^{\infty} Se^{-\mu S}\ dS$$

integrating by parts we have

$$E(S) = \mu e^{\mu S_0}\ [-\frac{S}{\mu}\ e^{-\mu S}\ \Big|_{S_0}^{\infty} - \int_{S_0}^{\infty} -\frac{1}{\mu}\ e^{-\mu S}\ dS]$$

$$= \mu e^{\mu S_0}\ [-\frac{S}{\mu}\ e^{-\mu S}\Big|_{S_0}^{\infty} - \frac{1}{\mu^2}\ e^{-\mu S}\Big|_{S_0}^{\infty}$$

$$= \mu e^{\mu S_0}[\frac{S_0}{\mu}\ e^{-\mu S_0} + \frac{1}{\mu^2}\ e^{-\mu S_0}]$$

$$E(S) = S_0 + \frac{1}{\mu}$$

$$\frac{1}{\mu} = E(S)-S_0$$

but if $E(S)=\overline{S}_t$

$$\mu = \frac{1}{\overline{S}_t-S_0}$$

but if we had

$$F(q) = \text{Prob } (S \leq q)$$

we know that

$$F(q) = \mu \int_{S_0}^{q} e^{-\mu(S-S_0)} dS = \mu e^{\mu S_0} \int_{S_0}^{q} e^{-\mu S} dS$$

$$F(q) = \mu \frac{e^{+\mu S_0}}{(-\mu)} (e^{-\mu q} - e^{-\mu S_0})$$

$$F(q) = 1 - e^{+\mu(S_0 - q)}$$

the reliability measure R(q) is defined as

$$R(q) = 1 - F(q)$$

then

$$R(q) = e^{+\mu(S_0-q)} = e^{-\mu(q-S_0)}$$

Therefore for the t^{th} constraint we have:

$$\text{Prob } (S_t \leq P_t + I_{t-1}) \geq u_t$$

then

$$1 - R(P_t + I_{t-1}) \geq u_t$$

$$1 - e^{-\mu(P_t + I_{t-1} - S_0)} \geq u_t$$

$$1 - u_t \geq e^{-\mu(P_t + I_{t-1} - S_0)}$$

taking logarithms to both side of the last expression we have:

$$\ln(1-u_t) \geq -\mu(P_t+I_{t-1}-S_0)$$

dividing by $-\mu$ we have:

$$-\frac{1}{\mu}\ln(1-u_t) \leq P_t + I_{t-1} - S_0$$

rearrange the terms to get

$$P_t + I_{t-1} + \frac{1}{\mu}\ln(1-u_t) \geq S_0$$

If we make

$$y_t \equiv -\ln(1-u_t)$$

and

$$\lambda \equiv \frac{1}{\mu}$$

that is

$$\lambda = \overline{S}_t - S_0$$

then,

$$P_t + I_{t-1} - \lambda y_t \geq S_0$$

The use of exponential distribution is motivated by its wide applications in statistical theory of reliability (142, 73) and the fact that it retains the constraint linearity of the transformed model, after the reliability measures are incorporated. The exponential distribution provides a limiting distribution in the class of distributions with nonnegative domains, under certain conditions (9), as the normal distribution under certain conditions provides a limiting distribution under the central limit theorem for a wide class of random variables. If we assume that the random variable S_t's are statistically independent and it is assumed that the system reliability R for a mathematical programming model as a whole must be an index combining the component reliabilities R_t, then one possible measure of system reliability is: (113, 112, 115)

$$R \equiv 1 - \prod_{t=1}^{T} R_t \equiv 1 - \prod_{t=1}^{T} (1-u_t) \qquad [4.1.1]$$

This measure R of system reliability is based on two main assumptions (115):

(i) The quadratic programming system consisting of t chance constraints is assumed if and only if each t^{th} constraint is feasible and reliable up to the tolerance level u_t $(t=1,\ldots,T)$

(ii) The system reliability measure R in [4.1.1] is bounded in the interval [0,1] as also the reliability component R_t. This concept of R in [4.1.1] will be used in our application subject to specified lower and upper bound restrictions on each u_t. This view of R preserves the quadratic structure of our program.

Note that [4.1.1] is one of several possible forms of specification of the system reliability for quadratic programming problems.

We may now present a way of incorporating the system reliability R in [4.1.1] into our original quadratic programming problem of probabilistic linear constraints. It is possible to associate a monotonic pay-off function with R and hence a utility function $U=U(R,-z)$ of the problem maker with reliability R and costs $(z=p'x+x'cx)$ as two arguments satisfying the usual conditions

$$\frac{\partial u}{\partial(-z)} > 0 \qquad \frac{\partial u}{\partial R} > 0 \qquad \frac{dR}{d(-z)} < 0 \qquad [4.1.2]$$

which imply risk aversion and guarantee the attainment of the maximal value of the U function in the cases where it is concave in its arguments. One possible choice of the utility function satisfying the conditions [4.1.2] is

$$U = \omega_1\{\sum_{t=1}^{T} [c_1-c_6)w_t+c_2(w_t-w_{t-1})^2+c_3(P_t-c_4w_t)^2$$
$$+c_5P_t+c_7(I_t-c_8-c_9\bar{S}_t)^2]\}-\omega_2\sum_{t=1}^{T}-\ln(1-u_t)$$

where ω_1 and ω_2 are scalar nonnegative constant weights assumed known such that $\omega_1+\omega_2=1$.

In the present case the transformed programming problem which incorporates the system reliability may be presented as follows:

$$\text{Min } \omega_1\{\sum_{t=1}^{T}[(c_1-c_6)w_t+c_2(w_t-w_{t-1})^2+c_3(P_t-c_4w_t)^2$$

$$+c_5P_t+c_7(I_t-c_8-c_9\bar{S}_t)^2]-\omega_2\sum_{t=1}^{T}y_t$$

subject to

$$w_t \geq w_0$$

$$P_t+I_{t-1}-\lambda y_t \geq S_0$$

$$y_t \geq y^*$$

$$y_t \leq y_* \qquad [t=1,\ldots,T]$$

$$y^*, y_* \geq 0$$

where

w_0 is a preassigned minimum labor force level

y^* is an upper bound of variable y (number $y_t=-\ln(1-u_t)$ and it is supposed to be known

y_* is a lower bound of variable y and also known.

Our problem can be written as

$$\text{Min } \omega_1[(c_1-c_6)\sum_{t=1}^{12}w_t+(2c_2+c_3c_4^2)\sum_{t=1}^{11}w_t+(c_2+c_3c_4^2)w_{12}^2$$

$$-2c_2\sum_{t=1}^{11}w_tw_{t+1}-2w_0c_2w_1$$

$$-2c_3c_4\sum_{t=1}^{12}P_tw_t+c_5\sum_{t=1}^{12}P_t+c_3\sum_{t=1}^{12}P_t^2+c_7\sum_{t=1}^{12}I_t^2$$

$$-2c_7(c_8+c_9\overline{S}_t)\sum_{t=1}^{12}I_t$$

$$(+\ 24c_7c_8c_9\overline{S}_t+12c_7c_8^2+c^2w_0^2)]-\omega_2\sum_{t=1}^{12}y_i$$

subject to

$$w_t \geq w_0$$

$$P_t+I_{t-1}-\lambda y_t\geq S_0$$

$$y_t\geq y^*$$

$$y_t\leq y_* \quad \text{for } t=1,\ldots,12$$

$$y^*,y_*\geq 0$$

This problem is a quadratic programming problem of the following form

$$\text{Min } p'x - \frac{1}{2}x'cx$$

subject to

$$Ax\leq b$$

that is a quadratic objective function subject to linear constraints, where

$$c = \begin{bmatrix} \underline{w}_1 & \underline{p}_{12} & 0 & 0 \\ \underline{p}_{12} & \underline{p}_1 & 0 & 0 \\ 0 & 0 & \underline{I}_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad x = \begin{bmatrix} \underline{w} \\ \underline{p} \\ \underline{I} \\ \underline{y} \end{bmatrix}$$

$$p = \begin{bmatrix} w_p \\ p_p \\ I_p \\ y_p \end{bmatrix} \quad ; \quad b = \begin{bmatrix} w_b \\ I_b \\ y_b{}^* \\ y_{*b} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \underline{w}^1 & 0 & 0 & 0 \\ 0 & \underline{p}^1 & \underline{I}^1 & \underline{y}^1 \\ 0 & 0 & 0 & \underline{y}^2 \\ 0 & 0 & 0 & \underline{y}^3 \end{bmatrix}$$

Let us define our submatrices and reactors as

$$\underset{(12\text{x}12)}{\underline{w}_1} = \begin{bmatrix} 4c_2+2c_4^2c_3 & -2c_2 & 0 & 0 & & \\ -2c_2 & 4c_2+2c_4^2c_3 & -2c_2 & 0 & & \\ 0 & & 4c_2+2c_4^2c_3 & -2c_2 & & \\ & & & & 4c_2+2c_4^2c_3 & -2c_2 \\ 0 & 0 & & 0 & & 2c_2+2c_4^2c_3 \end{bmatrix}$$

$$\underset{(12\text{x}12)}{\underline{p}_{12}} = \begin{bmatrix} -2c_4c_3 & 0 & & & \\ 0 & -2c_4c_3 & 0 & & \\ 0 & 0 & -2c_4c_3 & & \\ & & & & 0 \\ & & & 0 & -2c_4c_3 \end{bmatrix} ;$$

$$\underset{(12\times 12)}{\underline{p_1}} = \begin{bmatrix} 2c_3 & 0 & & \\ 0 & 2c_3 & & \\ & & & 0 \\ & & 0 & 2c_3 \end{bmatrix};$$

$$\underset{(12\times 12)}{\underline{I_1}} = \begin{bmatrix} 2c_7 & 0 & & \\ 0 & 2c_7 & & \\ & & & 0 \\ & & 0 & 2c_7 \end{bmatrix};$$

$$\underset{(12\times 1)}{\underline{w}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_{12} \end{bmatrix}; \quad \underset{(12\times 1)}{\underline{p}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{12} \end{bmatrix}; \quad \underset{(12\times 1)}{\underline{I}} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_{12} \end{bmatrix}; \quad \underset{(12\times 1)}{\underline{y}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{12} \end{bmatrix};$$

$$\underset{(12x1)}{\underline{w_p}} = \begin{bmatrix} c_1-c_6-2c_2w_0 \\ c_1-c_6 \\ \vdots \\ c_1-c_6 \end{bmatrix}; \quad \underset{(12x1)}{\underline{p_p}} = \begin{bmatrix} c_5 \\ c_5 \\ \vdots \\ c_5 \end{bmatrix}; \quad \underset{(12x1)}{\underline{I_p}} = \begin{bmatrix} -2c_7(c_8+c_9\bar{S}_t) \\ \vdots \\ -2c_7(c_8+c_9\bar{S}_t) \end{bmatrix};$$

$$\underset{(12x1)}{\underline{y_p}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix};$$

$$\underset{(12x1)}{\underline{w_p}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}; \quad \underset{(12x1)}{\underline{I_p}} = \begin{bmatrix} -I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad \underset{(12x1)}{\underline{y}_b^*} = \begin{bmatrix} \ln(1-u_t) \\ \vdots \\ \ln(1-u_t) \end{bmatrix};$$

$$\underset{(12x1)}{\underline{y}_{*b}} = \begin{bmatrix} \ln(1-u_t) \\ \vdots \\ \ln(1-u_t) \end{bmatrix};$$

$$\underset{(12\times12)}{\underline{w}^1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & & \\ & & & & 1 \end{bmatrix};$$

$$\underset{(12\times12)}{\underline{p}^1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & & \\ & & & & 1 \end{bmatrix};$$

$$\underset{(12\times12)}{\underline{I}^1} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & & & \\ & & & 1 & 0 \end{bmatrix}; \quad \underset{(12\times12)}{\underline{y}^1} = \begin{bmatrix} -\lambda & & & \\ & -\lambda & & \\ & & & \\ & & & -\lambda \end{bmatrix}$$

$$\underline{y}^2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \\ & & 1 \end{bmatrix} \text{ and } \underline{y}^3 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & \\ & & -1 \end{bmatrix}$$

As we know the values of u_t must be between the interval [0,1] thus we have preassigned an upper bound of 0.999 to y^* and a lower bound of 0.5 to y_*. Then for

$$y_t = -\ln(1-u_t)$$

we have that:

<u>for a lower bound</u>

$$u_t \geq .5$$

multiplying by -1

$$-u_t \leq -.5$$

adding 1 to both sides

$$1-u_t \leq .5$$

taking logarithms of both sides

$$\ln(1-u_t) \leq \ln .5$$

$$\ln(1-u_t) \leq -.69315$$

multiplying both sides by -1

$$-\ln(1-u_t) \geq .69315$$

then we have

$$y_t \geq .69315$$

<u>for an upper bound</u>

$$u_t \leq .999$$

multiplying by -1

$$-u_t \geq -.999$$

adding 1 to both sides

$$1-u_t \geq .001$$

taking logarithms of both sides

$$\ln(1-u_t) \geq \ln .001$$

$$\ln(1-u_t) \geq -6.90776$$

multiplying both sides by -1

$$-\ln(1-u_t) \leq 6.90776$$

then we have

$$y_t \leq 6.90776$$

We have developed a programming system for the present IBM system 360 Model 65 of the I.S.U. computer center, using the Zorrilla program (125, 146) to handle the quadratic programming problem which uses the procedure developed by Van de Panne and Whinston (96, 95). The programming system is so time varying and needs only four cards; for more details see the Appendix A.

We have a problem to determine how large λ can be (i.e. how much it will be possible to vary the interval $(\bar{S}_t - S_o)$ in order to deal with a significant case; we might have used the theoretical development of Bhattacharya and Holla (15). They have studied the exponential distribution with the probability density function

$$f(x,\theta) = \frac{1}{\theta} e^{-x/\theta} \quad (x>0,\ \theta>0) \qquad [4.1.3]$$

as a model for life-lengths of materials which is most thoroughly explored in (42, 43, 41). This distribution has a number of desirable mathematical properties, but its usefulness is sharply limited due to the property of the constancy of failure rate. For life-lengths of materials for which this property is no longer valid, normal distribution sometimes proves to be a nice analytical tool. Further, there exists many life-test situations which can be described as a mixture of two or more distributions (101, 77). Such situations arise because of the heterogeneity of the material involved in the life-test. However, if continuous variations are taking place during the manufacturing process, which is more or less inevitable, the quality of the produced items will be so much heterogeneous that the method of mixtures mentioned above may not be adequate enough to describe a life-test situation involving these items. In our case we only have to consider such a situation in which the mean life involved in [4.1.3] is subject to continuous stochastic deviations with a maximum permissible limit $\delta(>0)$ on either side of the mean. Assuming that these deviations are uniformly distributed over the range mentioned above, a new model for life length of materials is obtained.

We must remember that we defined the hazard ratio as

$$z(t) = \frac{f(t)}{1-F(t)}$$

The main reason for defining the z(t) function is that it is often more convenient to work with than f(t), (123, 103). For example, in our case f(t) is an exponential, the most common failure density one deals with in reliability work. Then

$$f(t) = \frac{1}{\theta} e^{-t/\theta}$$

$$F(t) = 1-e^{-t/\theta}$$

$$1-F(t) = e^{-t/\theta}$$

$$z(t) = \frac{f(t)}{1-F(t)} = \frac{\frac{1}{\theta}e^{-t/\theta}}{e^{-t/\theta}} = \frac{1}{\theta} ,$$

Thus, an exponential failure density compounds to constant hazard functions as pointed out above.

Bhattacharya and Holla considered a probability density function $f(x,\lambda)$ of the same form as [4.1.3], where the parameter λ is itself a random variable, distributed uniformly over a range $[\theta-\delta,\theta+\delta]$ with $\theta>\delta>0$. Then using the result

$$\int_x^\infty e^{-t}t^{-1}dt = e^{-x}\phi[1;1;x]$$

where $\phi[a;b;x]$ is the well-known function due to Tricomi (44, p. 255) defined for any real $a>0$, it is obtained for the over all distribution of x. The following probability density function:

$$f(x) = \frac{1}{2\delta}\int_{\theta-\delta}^{\theta+\delta}\frac{1}{\lambda}e^{-x/\lambda}d\lambda$$

$$= \frac{1}{2\delta}\{\exp(-x/(\theta+\delta))\phi[1;1;\frac{x}{\theta+\delta}]$$

$$-\exp(-x/(\theta-\delta))\phi[1;1;\frac{x}{\theta-\delta}]\}$$

$$(x>0,\theta>\delta>0) \qquad [4.1.4]$$

When δ approaches zero the uniform distribution considered above degenerates into a single point θ with the whole probability mass concentrated at this point and we obtain the usual exponential distribution in the limit. It is easy to see mathematically

$$\lim_{\delta\to 0} f(x) = \frac{1}{\theta}\exp(-x/\theta) \qquad (x>0,\ \theta>0)$$

The moments of the distribution [4.1.4] may be defined as

$$\mu'_r = \frac{r!}{(r+1)!}\left[\frac{(\theta+\delta)^{r+1}-(\theta-\delta)^{r+1}}{2\delta}\right]$$

In particular the first two moments are

$$\mu'_1 = \theta$$

$$\mu'_2 = \frac{2}{3}(3\theta^2+\delta^2)$$

The cumulative distribution function corresponding to [4.1.4] is

$$F(y) = \int_0^y f(x)\,dx = \frac{1}{2\delta}\int_{\theta-\delta}^{\theta+\delta}(1-e^{-y/\lambda})\,d\lambda$$

Then the failure rate is given by

$$z(t) = \frac{f(t)}{1-F(t)} = \frac{\frac{1}{2\delta}\int_{\theta-\delta}^{\theta+\delta}\frac{1}{\lambda}\,e^{-t/\lambda}d\lambda}{1-\frac{1}{2\delta}\int_{\theta-\delta}^{\theta+\delta}(1-e^{-t/\lambda})\,d\lambda}$$

$$z(t) = \frac{\int_{\theta\ \delta}^{\theta\ \delta}\frac{1}{\lambda}\,e^{-t/\lambda}d\lambda}{\int_{\theta-\delta}^{\theta+\delta}e^{-t/\lambda}d\lambda}$$

unlike we show for the exponential distribution, this failure is not independent of t.

For estimation of the parameters we have to have a random sample $(x_1, x_2, \ldots, x_n)$ from the distribution [4.1.4] and by the usual procedure of moments we have

$$\hat{\theta} = \nu_1$$

$$\hat{\delta} = \left(\frac{3}{2}\nu_2 - 3\nu_1^2\right)^{\frac{1}{2}}$$

where

$$\nu_r = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

We have, taking a much simpler approach, in a sense of simulation we have evaluated problems for different λ (i.e. $\overline{S}_t - S_0$) we observe in Table 4.1.1 these values

Table 4.1.1. Parameter values

λ	$\overline{S}$	S_0	Table
50	500	450	4.1.3
200	500	300	4.1.4
250	500	250	4.1.5
300	1100	800	4.1.6

We had observed that we could get better cases for λ=200 that is we get a little more variability in the probability. In Table 4.1.3, 4.1.4 and 4.1.5 we can observe how the change in labor force, production and inventory level are adjusting to the probability level to achieve a production planning which it is desired.

In the present case we faced one other problem: How large will be our production planning horizon (that is, how large could T be?) This point opens a very interesting question in the class of control theory problems (33, 4). Here we just solve problems with T up to 12 periods and we use as a criterion the stability of the solutions (i.e. steady-state values of our instrument variables).

We observe in Table 4.1.2 and in Figures 4.1.1, 4.1.2, 4.1.3 and 4.1.4, when T=12, a steady-state of values in instrument variables is achieved at T=6. Therefore, we only need to study production planning up to the sixth period in all cases.

As the cost coefficients and actual production, inventory and labor force had been coded to protect the company, we are dealing here with hypothetical cases of sales, in other words the average sale and minimum sales are not very representative about the labor force which always after certain values of t less than T goes to the minimum value given as restrictions.

We can also say that initial labor force and minimum labor restriction play a crucial role in our production planning problem, as they are shown in the tables below.

We also point out the influence of initial inventory in the probability level. We can see that the optimal inventory

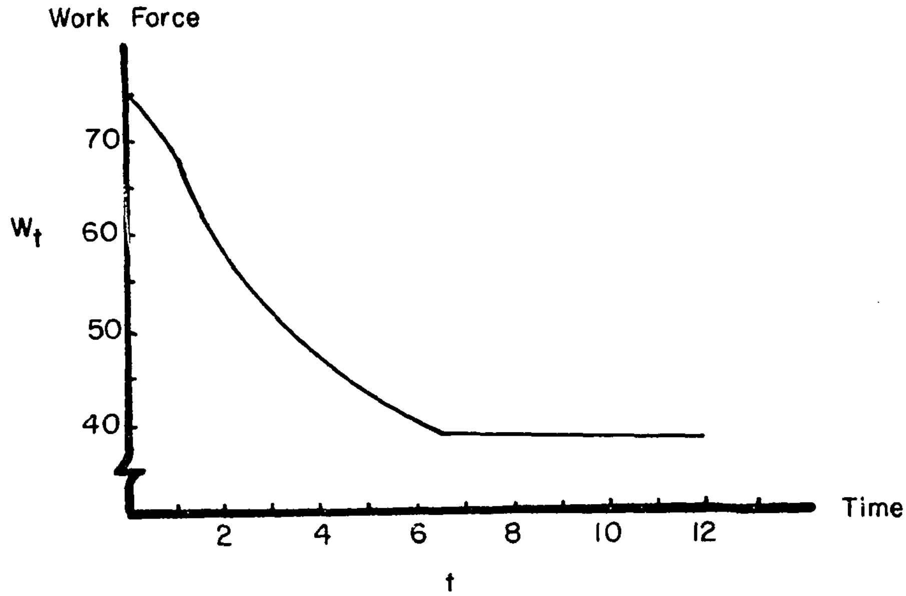

Figure 4.1.1. Work Force

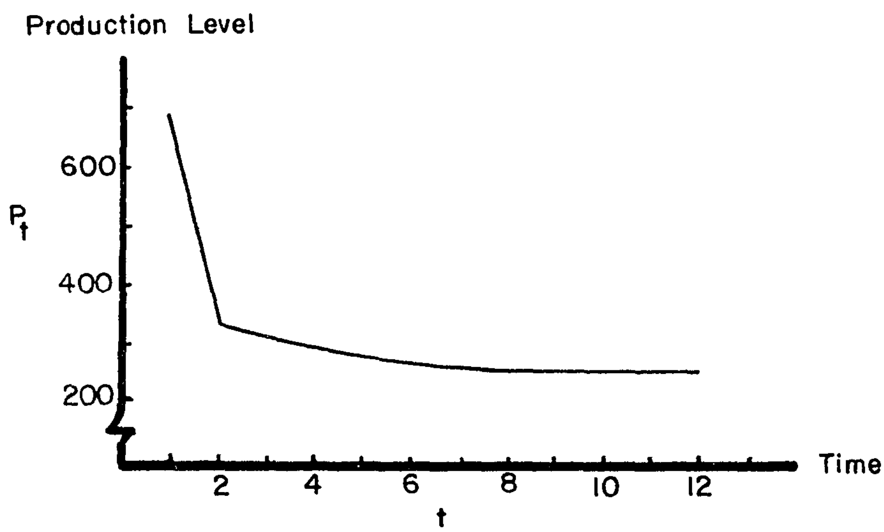

Figure 4.1.2. Production

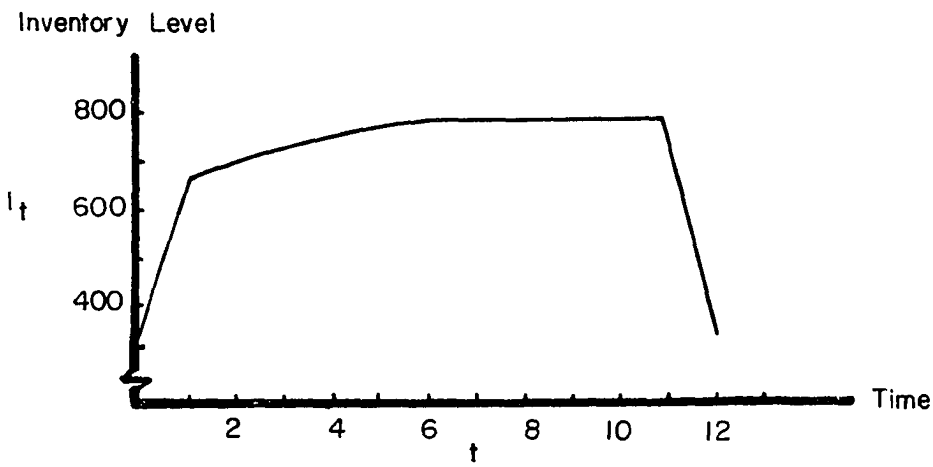

Figure 4.1.3. Inventory

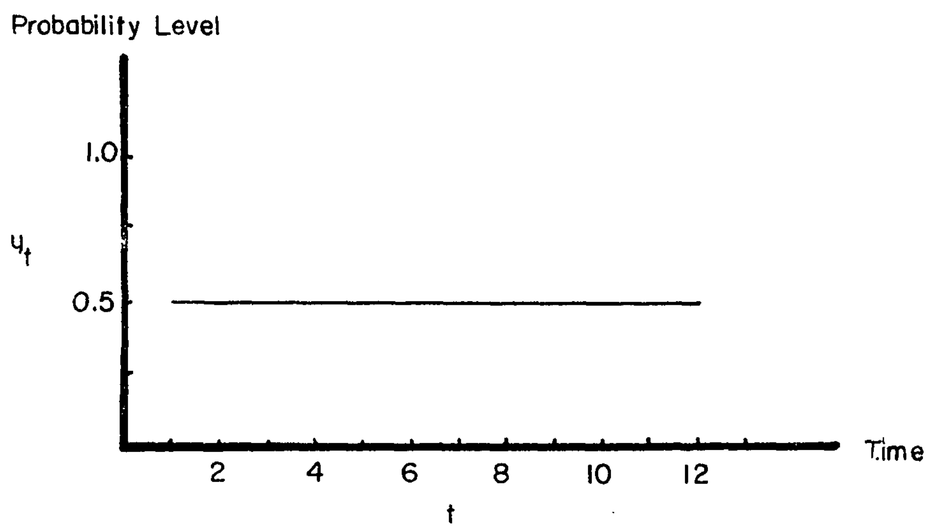

Figure 4.1.4. Probability Level

Table 4.1.2. Optimal solutions of the quadratic program (exponential reliability)

t=12 $\omega_1 = .8$ $w_0 = 75$ $\bar{S} = 1100$ $I_0 = 320$

$\omega_2 = .2$ $w_i \geq 40$ $S_0 = 800$ Total cost=882,642.36

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	70	687.945	653.306	.69315023	.5
2	61	354.640	683.053	.69315023	.5
3	53	324.892	705.736	.69315023	.5
4	48	302.210	722.016	.69315023	.5
5	44	285.930	732.368	.69315023	.5
6	41	275.578	737.094	.69315023	.5
7	40	270.851	737.094	.69315023	.5
8	40	270.851	737.094	.69315023	.5
9	40	270.851	737.094	.69315023	.5
10	40	270.851	737.094	.69315023	.5
11	40	270.851	737.094	.69315023	.5
12	40	270.851	320.000	.69315023	.5

$u_i \leq .999$; $u_i \geq .5$

Table 4.1.3. Optimal solutions of the quadratic program (exponential reliability)

t=6 $\omega_1 = .8$ $w_0 = 90$ $S = 500$ $I_0 = 320$

$\omega_2 = .2$ $w_i \geq 75$ $\bar{S}_0 = 450$ Total cost = 158,611.04

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.378	320.03	4.1075573	.983
2	76	303.843	320.03	3.47745	.969
3	75	297.263	320.03	3.3458491	.904
4	75	297.263	320.03	3.3458491	.904
5	75	297.263	320.03	3.3458491	.904
6	75	297.263	320.03	3.3458491	.904

Table 4.1.4. Optimal solutions of the quadratic program (exponential reliability)

t=6 $\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 0$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 170,102.51

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	83	438.630	320.007	.69315023	.5
2	77	307.060	320.007	1.6353335	.805
3	75	297.253	320.007	1.5863018	.795
4	75	297.253	320.007	1.5863018	.795
5	75	297.253	320.007	1.5863018	.795
6	75	297.253	320.000	1.5863018	.795

Table 4.1.5. Optimal solutions of the quadratic program (exponential reliability)

$\underline{t=6}$ $\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 0$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 250$ Total cost = 158,605.83

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.367	320.006	1.6214675	.803
2	76	303.832	320.006	1.4953506	.776
3	75	297.253	320.006	1.4690329	.760
4	75	297.253	320.006	1.4690329	.760
5	75	297.253	320.006	1.4690329	.760
6	75	297.253	320.000	1.4690329	.760

Table 4.1.6. Optimal solutions of the quadratic program (exponential reliability)

$\underline{t=6}$ $\omega_1 = .8$ $w_0 = 75$ $\bar{S} = 1100$ $I_0 = 320$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 800$ Total cost = 430,435.26

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	77	687.945	596.598	.69315023	.5
2	75	411.347	596.598	.69315023	.5
3	75	411.347	596.598	.69315023	.5
4	75	411.347	596.598	.69315023	.5
5	75	411.345	596.598	.69315023	.5
6	75	411.345	596.598	.69315023	.5

Table 4.1.7. Optimal solutions of the quadratic program (exponential reliability)

t=6

$\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 0$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 170,102.51

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	83	438.630	320.007	.69315023	.5
2	77	307.059	320.007	1.6353335	.805
3	75	297.253	320.007	1.5863018	.795
4	75	297.253	320.007	1.5863018	.795
5	75	297.253	320.007	1.5863018	.795
6	75	297.253	320.000	1.5863018	.795

Table 4.1.8. Optimal solutions of the quadratic program (exponential reliability)

t=6

$\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 160$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 158,606.16

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.367	320.007	.97683777	.623
2	76	303.833	320.007	1.6191993	.802
3	75	297.253	320.007	1.5863018	.795
4	75	297.253	320.007	1.5863018	.795
5	75	297.253	320.007	1.5863018	.795
6	75	297.253	320.007	1.5863018	.795

Table 4.1.9. Optimal solutions of the quadratic program (exponential reliability)

$\omega_1 = .8 \quad w_0 = 90 \quad \bar{S} = 500 \quad I_0 = 480$

$\omega_2 = .2 \quad w_i \geq 75 \quad S_0 = 300 \quad$ Total cost = 158,606.16

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.368	320.01	2.57683	.924
2	76	303.832	320.01	1.619199	.802
3	75	297.253	320.01	1.586302	.795
4	75	297.253	320.01	1.586302	.795
5	75	297.253	320.01	1.586302	.795
6	75	297.253	320.00	1.586302	.795

Table 4.1.10. Optimal solutions of the quadratic program (exponential reliability)

$\omega_1 = .8 \quad w_0 = 90 \quad \bar{S} = 500 \quad I_0 = 320$

$\omega_2 = .2 \quad w_i \geq 75 \quad S_0 = 300 \quad$ Total cost = 158,606.16

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.368	320.01	1.776838	.831
2	76	303.833	320.01	1.619199	.802
3	75	297.253	320.01	1.586302	.795
4	75	297.253	320.01	1.586302	.795
5	75	297.253	320.01	1.586302	.795
6	75	297.253	320.00	1.586302	.795

size is 320.0 gallons. That is when the expression $c_7(I_0-c_8)$ will be equal to zero. Then for an initial inventory of zero the probability to achieve the production planning in the first period is very low i.e., 50% while for I_0 equal to 160 it is 62.3% in the same period, while for I_0 equal to 320, the probability rises to 83.1% and if we had an initial inventory equal to 480 the probability level for the first period will be 92.4%. We can observe in Tables 4.1.7, 4.1.8, 4.1.9, and 4.1.10 that our planning variables are almost insensitive for planning periods other than the first period.

Since the Lagrange multipliers are almost the same in most cases we only present in Table 4.1.11 those associated with the solution indicated in Table 4.1.10.

Table 4.1.11. Lagrange multipliers associated with the solution in Table 4.1.10

Code	Restriction	Value
R127	labor force in 3rd period	333.4237
R128	labor force in 4th period	452.8061
R129	labor force in 5th period	452.8061
R130	labor force in 6th period	452.8061
R131	production in 1st period	0.000999
R134	production in 4th period	0.000999
R135	production in 5th period	0.000999
R136	production in 6th period	0.000999

We see in Table 4.1.11 that the most binding restrictions are the labor force in periods 4, 5, and 6.

4.1.2. Comparison with chance constrained programming

We have seen in section 4.1.1 a system reliability model for production planning, here we will have a quadratic programming problem with chance constraints which develops a method of providing appropriate safety margin under chance constraints, by incorporating the distributional characteristics of the random variables of the problem e.g., the sale vector in our production planning model. In this method a tolerance level, in terms of probability measure, one for which probability constraint is preassigned by us and this set of tolerance is supposed to indicate the limit up to which constraint violations are permitted by our satisfying behavior.

If we assume that sales have an exponential distribution as

$$f(S;\mu,S_0) = \mu\ e^{-\mu(S-S_0)}$$

then our restriction

$$\text{Prob }(p_t+I_{t-1} \geq S_t) \geq u_t$$

will be

$$\text{Prob}(S_t \leq p_t + I_{t-1}) \geq u_t$$

$$F(t) = \text{Prob}(S\leq t) = 1-e^{-\mu(t-S_0)}$$

$$R(t) = 1-F(t) = e^{-\mu(t-S_0)}$$

Therefore

$$\text{Prob}(S_t \le p_t + I_{t-1}) \ge u_t$$

$$1-R(p_t + I_{t-1}) \ge u_t$$

$$1-u_t \ge e^{-\mu(p_t+I_{t-1}-S_0)}$$

taking logarithm

$$\ln(1-u_t) \ge -\mu(p_t+I_{t-1}-S_0)$$

then

$$p_t+I_{t-1}-S_0 \ge -\frac{1}{\mu}\ln(1-u_t)$$

$$p_t+I_{t-1} \ge S_0 - \frac{1}{\mu}\ln(1-u_t)$$

or

$$p_t+I_{t-1} \ge S_0 -\lambda \ln(1-u_t)$$

Now, we must instead preassign a value for u_t, whereas we had before defined $\ln(1-u_t)=-y_t$ as a new variable. We have solved several problems for u_t equal to .85, .90, .95 and .99. The results are in Tables 4.1.12, 4.1.13, 4.1.14, 4.1.15, 4.1.16, 4.1.17, and 4.1.18. The production variable is more sensitive to the increase in the assigned value of u_t and initial inventory.

Table 4.1.12. Optimal solution of chance constrained programming models

t=6

$\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 0$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 401,949.88

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	89	899.146	494.957	2.9957296	.95
2	81	404.189	513.277	2.9957296	.95
3	77	385.870	519.572	2.9957296	.95
4	75	379.574	519.572	2.9957296	.95
5	75	379.574	519.572	2.9957296	.95
6	75	379.574	320.000	2.9957296	.95

Table 4.1.13. Optimal solution of chance constrained programming models

t=6

$\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 0$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 237,309.42

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	86	679.423	351.151	1.8971202	.85
2	78	328.274	364.017	1.8971202	.85
3	75	315.407	364.017	1.8971202	.85
4	75	315.407	364.017	1.8971202	.85
5	75	315.407	364.017	1.8971202	.85
6	75	315.407	320.000	1.8971202	.85

Table 4.1.14. Optimal solution of chance constrained programming models

$\underline{t=6}$ $\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 320$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 280,226.71

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	85	579.146	505.132	2.9957296	.95
2	79	394.013	518.291	2.9957296	.95
3	75	380.855	519.572	2.9957296	.95
4	75	379.574	519.572	2.9957296	.95
5	75	379.574	519.572	2.9957296	.95
6	75	379.574	519.572	2.9957296	.95

Table 4.1.15. Optimal solution of chance constrained programming models

$\underline{t=6}$ $\omega_1 = .8$ $w_0 = 90$ $\bar{S} = 500$ $I_0 = 160$

$\omega_2 = .2$ $w_i \geq 75$ $S_0 = 300$ Total cost = 331,554.89

Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	87	739.146	500.450	2.9957296	.95
2	80	399.101	515.784	2.9957296	.95
3	76	383.362	519.572	2.9957296	.95
4	75	379.574	519.572	2.9957296	.95
5	75	379.574	519.572	2.9957296	.95
6	75	379.574	320.000	2.9957296	.95

Table 4.1.16. Optimal solution of chance constrained programming models

t=6	$\omega_1 = .8$	$w_0 = 90$	$\bar{S} = 500$	$I_0 = 320$	
	$\omega_2 = .2$	$w_i \geq 75$	$S_0 = 300$	Total cost = 170,398.61	
Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	359.424	358.097	1.8971202	.85
2	77	321.327	364.017	1.8971202	.85
3	75	315.407	364.017	1.8971202	.85
4	75	315.407	364.017	1.8971202	.85
5	75	315.407	364.017	1.8971202	.85
6	75	315.407	364.017	1.8971202	.85

Table 4.1.17. Optimal solution of chance constrained programming models

t=6	$\omega_1 = .8$	$w_0 = 90$	$\bar{S} = 500$	$I_0 = 320$	
	$\omega_2 = .2$	$w_i \geq 75$	$S_0 = 300$	Total cost = 612,406.18	
Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	90	901.034	709.713	4.60517	.99
2	84	511.321	727.490	4.60517	.99
3	80	493.544	739.499	4.60517	.99
4	77	481.535	746.091	4.60517	.99
5	75	474.943	747.458	4.60517	.99
6	75	473.576	320.000	4.60517	.99

Table 4.1.18. Optimal solution of chance-constrained programming models

t=6	ω_1 = .8	w_0 = 90	$\bar{S}$ = 500	I_0 = 320	
	ω_2 = .2	$w_i \geq 75$	S_0= 300	Total cost = 199,980.58	
Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability level y_t	Probability level u_t
1	83	440.518	412.669	2.3029995	.90
2	77	347.850	421.429	2.3029995	.90
3	75	339.090	421.429	2.3029995	.90
4	75	339.090	421.429	2.3029995	.90
5	75	339.090	421.429	2.3029995	.90
6	75	339.090	421.429	2.3029995	.90

If we assume that sales are normally and independently distributed as

$$f(S;m,\sigma) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{(S-m)^2}{2\sigma^2}}$$

Then our restriction

$$\text{Prob}(p_t+I_{t-1} \geq S_t) \geq u_t$$

will be transforms as

$$\text{Prob}(S_t \leq p_t+I_{t-1}) \geq u_t$$

$$\text{Prob}\left(\frac{S_t-m}{\sqrt{\sigma^2}} \leq \frac{p_t=I_{t-1}-m}{\sqrt{\sigma^2}}\right) \geq u_t$$

Then

$$F\left(\frac{p_t+I_{t-1}-m}{\sqrt{\sigma^2}}\right) \geq u_t$$

and

$$\frac{p_t+I_{t-1}-m}{\sqrt{\sigma^2}} \geq F^{-1}(u_t)$$

finally

$$p_t+I_{t-1}-m \geq \sigma F^{-1}(u_t)$$

$$p_t+I_{t-1} \geq m + \sigma F^{-1}(u_t)$$

When a normal distribution is assumed for the sales the mean μ and standard deviation σ are sufficient to fully determine the distribution function. As we have used in our earlier problems an exponential distribution with two parameters, we have the following estimates of the normal parameters m and S on the basis of our earlier results.

$$\mu^{-1} = S$$

$$\mu^{-1} = \bar{S}_t - S_0$$

$$S = 500-300 = 200$$

$$S_0 = m-S$$

then

$$m = S_0+S$$

$$m = 300+200 = 500$$

In Table 4.1.19 we assume that $F^{-1}(u_t)=-1.65$ that is the value of u_t is equal to 95%. If we compare these results with those given in Table 4.1.14 we observe that the normal assumption gives us lower results than the exponential distribution. This is because the exponential distribution gives higher ordinate values at lower abscissa values.

In Figure 4.1.5 we show a curve of substitution between cost and system reliability for several exponential CCP cases.

Table 4.1.19. Optimal solution of chance-constrained programming model (Normal case)

t=6	$\omega_1 = 1$	$w_0 = 90$	$\bar{m} = 500$	$I_0 = 320$	
	$\omega_2 = 0$	$w_i \geq 75$	S = 200	Total cost = 265,766.81	
Time t	Work Force w_t	Production level p_t	Inventory level I_t	Reliability y_t	Probability level u_t
1	82	335.36	320	1.65	.95
2	76	303.83	320	1.65	.95
3	75	297.25	320	1.65	.95
4	75	297.25	320	1.65	.95
5	75	297.25	320	1.65	.95
6	75	297.25	320	1.65	.95

Finally in Table 4.1.20 we have shown the Lagrange multipliers associated with the solution in Table 4.1.12 which indicates the most binding restriction to be the lower bound reliability for the first period.

Table 4.1.20. Lagrange multipliers associated with the solution in Table 4.1.12

Code	Restrictions	Value
R128	labor force in 4th period	142.0903
R129	labor force in 5th period	303.4436
R130	labor force in 6th period	303.4436
R131	production in 1st period	167.5241
R132	production in 2nd period	23.0944
R133	production in 3rd period	25.5126
R134	production in 4th period	26.3436
R135	production in 5th period	26.3436
R136	production in 6th period	26.3436
R137	lower bound reliability for the 1st period	33504.626
R138	lower bound reliability for the 2nd period	4618.6862
R139	lower bound reliability for the 3rd period	5102.3152
R140	lower bound reliability for the 4th period	5268.5210
R141	lower bound reliability for the 5th period	5268.5210
R142	lower bound reliability for the 6th period	5268.5210

4.2. Application of Multi-Period Investment under Uncertainty: Implication of Decisions Rules

Näslund (89, 90) considered a model of rational investment in the stock market. The model itself abstracts from a more general problem of resource carryover under risk. He assumes that a person has a known income stream. Furthermore, he assumes that a person has decided how to allocate his consumption up to a specified horizon. Each period the difference between income and consumption is made available for investment in the stock market. His goal is the

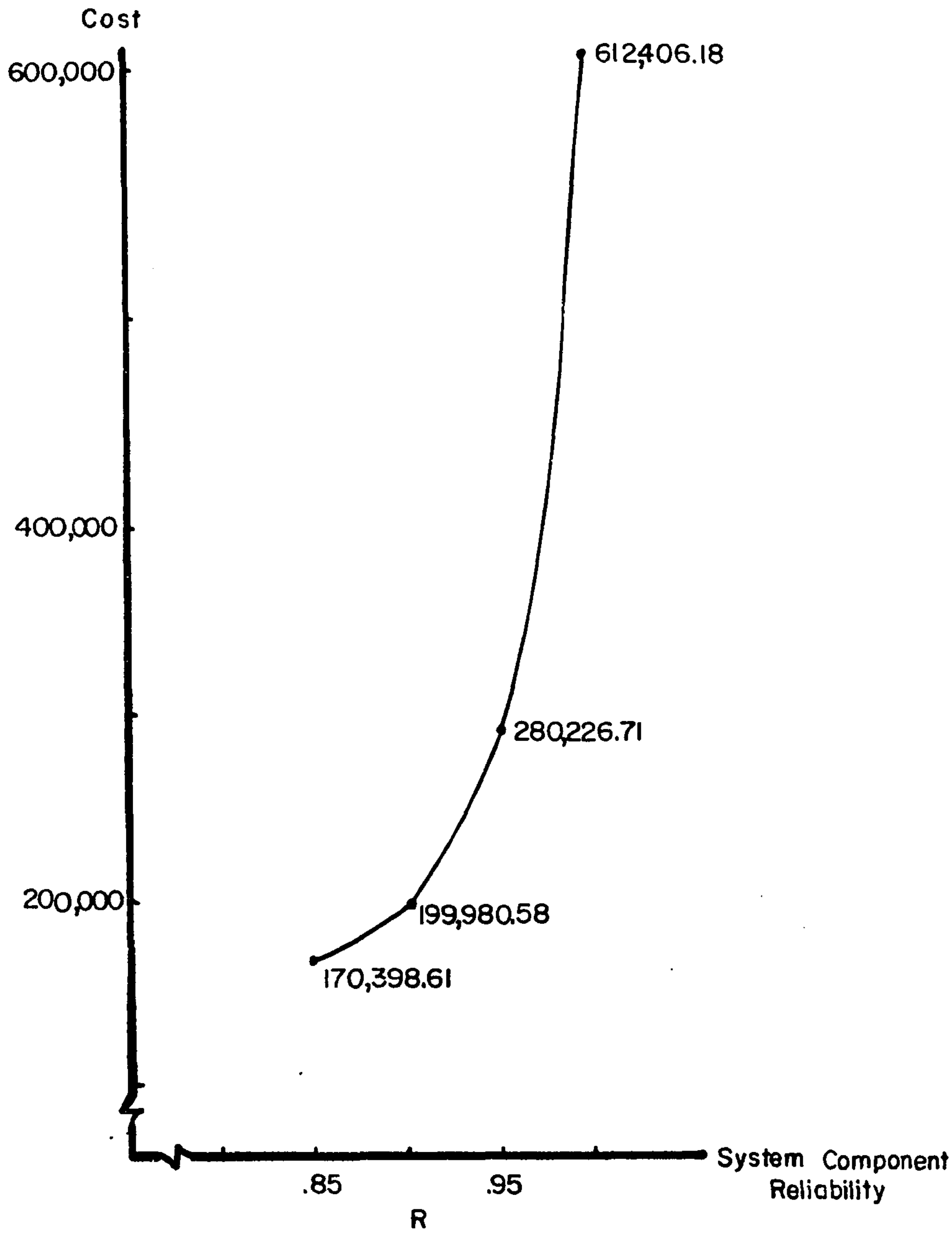

Figure 4.1.5. Curve of Substitution between Cost and System Component Reliability. CCP (2 Parameter - Exponential)

maximization of his expected gain in the stock market at the end of the specified horizon subject to certain constraints in each period. Such as: the risk constraint which in effect sets a probabilistic limit on possible losses beyond a specified amount; and a capital constraint which probabilistically stipulates that invested capital should be below a limit which varies according to accumulated capital gains. Näslund assumes that the investor is not affected in his risk aversion by the actual outcomes of his investments.

The problem is formulated in the following way

$$\text{Max } E\{\sum_{i=1}^{n} x_i \frac{p_i - p_{i-1}}{p_{i-1}}\} \qquad [4.2.1]$$

subject to

$$\text{Prob } \{x_i \frac{p_i - p_{i-1}}{p_{i-1}} \geq L_i\} \geq \alpha_i$$

$$\text{Prob } \{x_i \leq k_i + \sum_{j=2}^{i} x_{j-1} \frac{p_{j-1} - p_{j-2}}{p_{j-2}}\} \geq \eta_i$$

where

x_i is the accumulated amount in dollars invested in stocks

p_i is the stock price or group of stocks price in period i

L_i is the maximum loss that the person is willing to take in period i

α_i is the risk prescribed for period i

k_i is the capital accumulated that the person can use for investment either in cash or stocks apart from the returns on earlier investments

η_i is the risk level for capital constraint in period i

It is assumed that $(p_i - p_{i-1})/p_{i-1}$ is approximately normally distributed. Osborne has made such assumptions in (92).

In solving the above model Näslund pointed out that consideration must be given to the fact that certain data which are presently unavailable i.e., the stock prices, will be varying in the future, when actual decisions are made. For this reason he does not solve directly for the x's, but a decision rule of the following form

$$x_i = \sum_{j=2}^{i} \beta_j [1 + \frac{p_{j-1} - p_{j-2}}{p_{j-2}}] + \gamma_i \qquad [4.2.2]$$

is introduced in the model replacing x_i and he had to determine as a solution β's and γ's. Since x_k in period k is a function of known past prices at that time, the decision rule can be interpreted as the demand of a particular person to hold funds inverted in the stock market.

Näslund only considers a three year horizon in order to limit the numerical calculation. We will consider the same case and data in order to compare our results with his.

Therefore the problem can then be written as:

$$\text{Max } E\{\gamma_1 \frac{p_1-p_0}{p_0} + [\beta_2(1+ \frac{p_1-p_0}{p_0})+\gamma_2]\frac{p_2-p_1}{p_1}$$

$$+ [\beta_3(1+ \frac{p_2-p_1}{p_1})+\gamma_3] \frac{p_3-p_2}{p_2}\}$$

subject to loss constraints i.e., the maximum loss that the person is willing to take in period i

$$\text{Prob } (\gamma_1 \frac{p_1-p_0}{p_0} \geq L_1) \geq \alpha_1$$

$$\text{Prob } \{[\beta_2(1+ \frac{p_1-p_0}{p_0} + \gamma_2] \frac{p_2-p_1}{p_1} \geq L_2\} \geq \alpha_2$$

$$\text{Prob } \{[\beta_3(1+ \frac{p_2-p_1}{p_1})+ \gamma_3] \frac{p_3-p_2}{p_2} \geq L_3\} > \alpha_3$$

Capital constraints i.e. the capital accumulated that the person can use for investment either in **cash** or stocks apart from the return on early investment

$$\gamma_j \leq k_1$$

$$\text{Prob } \{\beta_2[1+ \frac{p_1-p_0}{p_0}] + \gamma_2 \leq k_2 + \gamma_1 \frac{p_1-p_0}{p_0} \geq \eta_2$$

$$\text{Prob } \{\beta_3[1+ \frac{p_2-p_1}{p_1}] + \gamma_3 \leq k_3 + \gamma_1 \frac{p_1-p_0}{p_0}$$

$$+ [\beta_2(1+ \frac{p_1-p_0}{p_0}) + \gamma_2] \frac{p_2-p_1}{p_1} \geq \eta_3$$

In order to solve numerically, Näslund assigned the following values to the parameters

L_1 = -1300 (dollars)

L_2 = -1000

L_3 = -1000

$\alpha_1 = \alpha_2 = \alpha_3 = 95\%$

k_1 = 7000 (dollars)

k_2 = 5500

k_3 = 9000

$\eta_2 = 99.997\%$

$\eta_3 = 99\%$

μ = 0.05, mean of the distribution of $(p_i - p_{i-1})/p_{i-1}$

σ = 0.15, standard deviation of the distribution of $(p_i - p_{i-1})/p_{i-1}$

He inserts these values in the constraints and considers $[(p_j - p_{j-1})/p_{j-1}]^n$ small for $n \geq 2$.

The deterministic equivalents to the probabilistic constraints are derived by methods indicated before. He uses the following property of normal variates

$$\text{Prob } \{y \geq Q\} = \frac{1}{\sqrt{2\Pi}} \int_{(Q-\mu)/\sigma}^{\infty} \exp(-y^2/2)dy = \frac{1}{2} - F\left(\frac{Q-\mu}{\sigma}\right)$$

where $F(x)$ is the normal function

$$F(x) = \frac{1}{\sqrt{2\Pi}} \int_0^x \exp(-y^2/2)dy$$

Therefore the constraints can be written as follows

$$\frac{1}{2} - F\left(\frac{-1300-0.05\ \gamma_1}{0.15\ \gamma_1}\right) \geq 0.95$$

$$\frac{1}{2} - F\left(\frac{-1000-0.05(\beta_2+\gamma_2)}{0.15(\beta_2+\gamma_2)}\right) \geq 0.95$$

$$\frac{1}{2} - F\left(\frac{-1000-0.05(\beta_3+\gamma_3)}{0.15(\beta_3+\gamma_3)}\right) \geq 0.95$$

$$\gamma_1 \leq 7000$$

$$\frac{1}{2} - F\left(\frac{\beta_2+\gamma_2-5500-0.05(\gamma_1-\beta_2)}{0.15(\gamma_1-\beta_2)}\right) \geq 0.9997$$

$$\frac{1}{2} - F\left(\frac{\beta_3+\gamma_3-9000-0.05(\gamma_1+\beta_2+\gamma_2-\beta_3)}{0.15\sqrt{(\beta_2+\gamma_2-\beta_3)^2+\gamma_1^2}}\right) \geq 0.99$$

If we rearrange the equations we get

$$F\left(\frac{1300+0.05\ \gamma_1}{0.15\ \gamma_1}\right) \geq 0.45$$

$$F\left(\frac{1000+0.05(\beta_2+\gamma_2)}{0.15(\beta_2+\gamma_2)}\right) \leq 0.45$$

$$\gamma_1 \leq 7000$$

$$F\left(\frac{5500+0.05(\gamma_1-\beta_2)-\beta_2-\gamma_2}{0.15(\gamma_1-\beta_2)}\right) \geq 0.49997$$

$$F\left(\frac{9000.+0.05(\gamma_1+\beta_2+\gamma_2-\beta_3)-\beta_3-\gamma_3}{0.15\sqrt{(\beta_2+\gamma_2-\beta_3)^2+\gamma_1^2}}\right) \geq 0.49$$

Using the transformation that if $z=F(y)$ then $y=F^{-1}(z)$ we get

$$\frac{1300+0.05\ \gamma_1}{0.15\ \gamma_1} \geq F^{-1}(0.45) \equiv 1.65$$

$$\frac{1000+0.05(\beta_2+\gamma_2)}{0.15(\beta_2+\gamma_2)} \geq F^{-1}(0.45) \equiv 1.65$$

$$\frac{1000+0.05(\beta_3+\gamma_3)}{0.15(\beta_3+\gamma_3)} \geq F^{-1}(0.45) \equiv 1.65$$

$$\gamma_1 \leq 7000$$

$$\frac{5500+0.05\ \gamma_1-1.05\ \beta_2+\gamma_2}{0.15(\gamma_1-\beta_2)} \geq F^{-1}(0.49997) \equiv 4.0$$

$$\frac{9000+0.05(\gamma_1+\beta_2+\gamma_2)-1.05\ \beta_3-\gamma_3}{0.15\sqrt{(\beta_2+\gamma_2-\beta_3)^2+\gamma_1^2}} \geq F^{-1}(0.49) \equiv 2.33$$

Once again if we rearrange the constraints above, the problem takes the following form.

$$\text{Max } E\{\sum_{i=1}^{n} x_i \frac{p_i-p_{i-1}}{p_{i-1}}\}$$

subject to

$$\gamma_1 \leq 6500$$

$$\gamma_2 + \beta_2 \leq 5000$$

$$\gamma_3 + \beta_3 \leq 5000$$

$$0.55\ \gamma_1 + 0.45\ \beta_2 + \gamma_2 \leq 5500$$

$$0.35\sqrt{(\beta_2+\gamma_2-\beta_3)^2+\gamma_1^2} + 1.05\ \beta_3 + \gamma_3 - 0.05(\gamma_1+\beta_2+\gamma_2) \leq 9000$$

The reason that we have only five constraints here, is because it turns out that $\gamma_1 \leq 6500$ dominates $\gamma_1 \leq 7000$ in the sense that it is more constraining. Here we have a major departure from Näslund, because he neglects the last restriction and solves only the linear part of the problem, e.g. Näslund solves only a linear programming problem. We solve instead the nonlinear programming problem and in Table 4.2.1 we can see the difference in solutions we got

Table 4.2.1. Decision rule solutions

Variable	Linear solution (LP model)	Nonlinear solution (NLP model)
γ_1	5900	5909.1632
γ_2	0	0.21259
γ_3	5000	2696.9716
β_2	5000	4999.0663
β_3	0	2302.8111

For solving the nonlinear solution we have used SUMT (sequential unconstrained minimization technique) (51, 126, 52, 53, 49, 48, 17, 50, 82). According to the decision rule given in (4.2.2) we have the following form of investment decisions where x_j denotes the total amount of money held in stocks in period j.

$$x_1 = \gamma_1 \tag{4.2.3}$$

$$x_2 = \beta_2(1+ \frac{p_1-p_0}{p_0}) + \gamma_2 \tag{4.2.4}$$

$$x_3 = \beta_3(1+ \frac{p_2-p_1}{p_1}) + \gamma_3 \tag{4.2.5}$$

Now if we insert the value of Table 4.2.1 in equations 4.2.3, 4.2.4, 4.2.5 we have

Linear solution

$$x_1 = 5900$$
$$x_2 = 5000(1+ \frac{p_1-p_0}{p_0})$$
$$x_3 = 5000$$

Nonlinear solution

$$x_1 = 5909.1632$$
$$x_2 = 4999.0663(1 + \frac{p_1-p_0}{p_0}) + 0.21259$$
$$x_3 = 2302.8111(1 + \frac{p_2-p_1}{p_2}) + 2696.9716$$

We can see that solving a nonlinear programming formulation gives us a slightly different problem; the decisions that explicitly depend on the random variable $\frac{p_{j-1}-p_{j-2}}{p_{j-2}}$ are the decisions in the second and third period while in linear formulation the random variable is important only in the second period. In Table 4.2.2 we compare the problems for a price increase of 5% or 10% between period 0 to 1 and 1 to 2.

Table 4.2.2. Price variation in decision rules[a]

Variable	Linear solution 5%	Linear solution 10%	Nonlinear solution 5%	Nonlinear solution 10%
x_1	5900	5900	5909.16	5909.16
x_2	5250	5500	5249.23	5499.19
x_3	5000	5000	5114.92	5230.06
z[b]	807.5	820.0	813.67	831.92

[a]These values are rounded.

[b]z denotes the value of the objective function of problem [4.2.1].

We can observe that there exists more variability in the value of the objective function of the nonlinear formulation due to changes in the random variable $\frac{p_{j-1}-p_{j-2}}{p_{j-2}}$.

If we do not want to use a decision rule as given in Equation [4.2.2], then we solve an optimization problem in nonlinear programming. We have then the following problem:

$$\text{Max } E(x_1 \frac{p_1-p_0}{p_0} + x_2 \frac{p_2-p_1}{p_1} + x_3 \frac{p_3-p_2}{p_2}) \qquad [4.2.6]$$

subject to

$$\text{Prob}(x_1 \frac{p_1-p_0}{p_0} \geq -1300) \geq .95 \qquad [4.2.7]$$

$$\text{Prob}(x_2 \frac{p_2-p_1}{p_1} \geq -1000) \geq .95 \qquad [4.2.8]$$

$$\text{Prob}(x_3 \frac{p_3-p_2}{p_2} \geq -1000) \geq .95 \qquad [4.2.9]$$

$$x_1 \leq 7000$$

$$\text{Prob}(x_2 \leq 5500 + x_1 \frac{p_1-p_0}{p_0} \geq .99997 \qquad [4.2.10]$$

$$\text{Prob}(x_3 \leq 9000 + x_1 \frac{p_1-p_0}{p_1} + x_2 \frac{p_2-p_1}{p_1}) \geq .99 \qquad [4.2.11]$$

as we know $\frac{p_i-p_{i-1}}{p_{i-1}}$ is normal and independently distributed with mean μ=.05 and standard deviation σ=.15. Therefore the objective function [4.2.6] will be

$$\text{Max } .05\ x_1 + .05\ x_2 + .95\ x_3 \qquad [4.2.12]$$

For restriction [4.2.7], [4.2.8] and [4.2.9] we can make the following simplifications:

The probabilities in those restrictions can then be transformed by simple subtraction and division as follows.

For restriction [4.2.7]

$$\text{Prob}\left(\frac{x_i\frac{p_i-p_{i-1}}{p_{i-1}} - \mu x_i}{\sqrt{\sigma^2 x_i^2}} \geq \frac{L_i-\mu x_i}{\sqrt{\sigma^2 x_i^2}}\right) \geq \alpha \qquad [4.2.13]$$

The left hand side of the argument of the probability expression is found to be a normalized random variable with zero mean and unit variance. Hence the probabilistic condition is replaced by

$$1 - F\left(\frac{L_i-\mu x_i}{\sigma x_i}\right) \geq \alpha$$

where

$$F(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

Then

$$F\left(\frac{L_i-\mu x_i}{\sigma x_i}\right) \leq 1-\alpha$$

therefore

$$L_i-\mu x_i \geq \sigma x_i F^{-1}(1-\alpha)$$

and

$$L_i-[\mu+\sigma F^{-1}(1-\alpha)]x_i \geq 0 \qquad [4.2.14]$$

For restriction [4.2.10] we use the notation $\frac{p_i-p_{i-1}}{p_{i-1}} \equiv \beta_i$ and $\bar{\beta}_i = E\left(\frac{p_i-p_{i-1}}{p_{i-1}}\right)$, $\sigma_{\beta_i}^2 = \text{Var}\left(\frac{p_i-p_{i-1}}{p_{i-1}}\right)$ and note that

$$\text{Prob}(x_2 \leq k_2 + x_1\beta_1) \geq \eta_2$$

which we rewrite as

$$\text{Prob}(x_1\beta_1 \geq x_2-k_2) \geq \eta_2$$

so

$$\text{Prob}\left(\frac{x_1\beta_1-x_1\bar{\beta}_1}{\sqrt{\sigma^2_{\beta_1}x_1^2}} \geq \frac{x_2-k_2-x_1\bar{\beta}_1}{\sqrt{\sigma_{\beta_1}x_1^2}}\right) \geq \eta_2$$

for the same reason given for expression [4.2.13] we have

$$1-F\left(\frac{x_2-k_2-x_1\bar{\beta}_1}{\sigma_{\beta_1}x_1}\right) \geq \eta_2$$

$$F\left(\frac{-x_2+k_2+x_1\bar{\beta}_1}{\sigma_{\beta_1}x_1}\right) \leq 1-\eta_2$$

and

$$-x_2+k_2+x_1\bar{\beta}_1 \geq F^{-1}(1-\eta_2)\sigma_{\beta_1}x_1$$

collecting terms

$$[-\bar{\beta}_1+\sigma_{\beta_1}F^{-1}(1-\eta_2)]x_1-x_2 \leq k_2 \qquad [4.2.15]$$

For restriction [4.2.11] we have the same procedure as follows:

we can rewrite [4.2.11] as

$$\text{Prob}(x_1\beta_1+x_2\beta_2 \geq x_3-k_3) \geq \eta_3$$

so we have

$$\text{Prob}\left(\frac{x_1\beta_1+x_2\beta_2-x_1\overline{\beta}_1-x_2\overline{\beta}_2}{\sqrt{\sigma_{\beta_1}^{2}x_1^2+\sigma_{\beta_1}^{2}x_2^2}} \geq \frac{x_3-k_3-x_1\overline{\beta}_1-x_2\overline{\beta}_2}{\sqrt{\sigma_{\beta_1}^{2}x_1^2+\sigma_{\beta_2}^{2}x_2^2}}\right) \geq \eta_3$$

as we already know $\overline{\beta}_1=\overline{\beta}_2=\overline{\beta}$ and $\sigma_{\beta_2}^{2}=\sigma_{\beta_2}^{2}=\sigma^2$ then we have

$$1-F\left(\frac{x_3-k_3-x_1\overline{\beta}-x_2\overline{\beta}}{\sigma\sqrt{x_1^2+x_2^2}}\right) \geq \eta_3$$

$$F\left(\frac{-x_3+k_3+x_1\overline{\beta}+x_2\overline{\beta}}{\sigma\sqrt{x_1^2+x_2^2}}\right) \leq 1-\eta_3$$

therefore

$$-\overline{\beta}x_1-\overline{\beta}x_2+x_3+\sigma F^{-1}(1-\eta_3)\sqrt{x_1^2+x_2^2} \leq k_3 \qquad [4.2.16]$$

Summarizing expressions [4.2.12], [4.2.14], [4.2.15], and [4.2.16] in the form fashionable to SUMT we will have the following model:

$$\text{Max } .05\,x_1 + .05\,x_2 + .05\,x_3 \qquad [4.2.17]$$

subject to

$$\begin{array}{llllll}
1300 + (.05-.15\,\psi)x_1 & & & & & \geq 0 \\
1000 & & + (.05-.15\,\psi)x_2 & & & \geq 0 \\
1000 & & & & + (.05-.15\,\psi)x_3 & \geq 0 \\
7000 & & -x_1 & & & \geq 0 \\
5500 + (.05-.15\,\Psi)x_1 & & & -x_2 & & \geq 0 \\
-.15\phi\sqrt{x_1^2+x_2^2} + 9000 & +.05x_1 & & +.05x_2 & -x_3 & \geq 0
\end{array}$$

where

$$\psi = F^{-1}(\alpha-\tfrac{1}{2})$$

$$\Psi = F^{-1}(\eta_2-\tfrac{1}{2})$$

$$\phi = F^{-1}(\eta_3-\tfrac{1}{2})$$

In our case ψ=1.65, Ψ=4., ϕ=2.33

Solving the nonlinear programming problem [4.2.17] we have the following result

x_1 = 6581.36

x_2 = 3528.65

x_3 = 5024.71

value of objective function z=756.74

Note that this optimization model has only 3 variables, while the nonlinear decision rule problem has 5 variables which make the latter model more difficult to solve in terms of programming and computing time. However in the latter model we have some flexibility of decisions in the second and third periods, while in the other case we take investment decision for three periods once and for all. Table 4.2.3 gives us a general view of our problems.

4.2.1. Reliability approach

Our chance-constrained approach of a multi-period investment under uncertainty provides the basis of reliability

Table 4.2.3. Summary of problems

Decision rule problem Näslund solution (Linear programming) (1)	Optimization problem Nonlinear programming solution (2)	Decision rule problem Nonlinear solution 10% price change (3)	Decision rule problem Nonlinear solution 5% price change (4)	System reliability problem at 99%
x_1=5900 (783225.0)[a]	x_1=6581.36 (974571.7376)	x_1=5909.163 (785659.6656)	x_1=5909.163 (785659.6656)	x_1=6999.975 (1102492.1250)
x_2=5500 (562500.0)	x_2=3528.65 (280155.8435)	x_2=5499.185 (680423.3024)	x_2=5249.232 (619974.823)	x_2=2440.3998 (133999.9016)
x_3=5000 (562500.0)	x_3=5024.71 (568073.4882)	x_3=5230.064 (615455.3125)	x_3=5114.923 (588654.8392)	x_3=8068.435 (1464741.9754)
z =820.0 (2026350.0)	z =756.74 (1822806.0693)	z =831.921 (2081538.281)	z =813.666 (1994289.328)	z =865.3399 (2701234.0020)
cv[b]=173.5975	cv=178.411	cv=173.4243	cv=173.559	cv=189.9304

[a]Figures in parenthesis indicate variance component $x_i^2\sigma_{c_i}^2$ at the optimal solution.

[b]cv=Coefficient of variation defined as $100X/\bar{x}$.

analysis. Here we can develop an operational measure of reliability for the nonlinear programming system. In this chance-constrained problem approach a tolerance level in terms of probability measures, one for each probabilistic constraint is preassigned by the decision-maker as we saw above and this set of tolerance measures is supposed to indicate the limit up to which constraint violations are permitted.
This view of chance-constraints problem allows the interpretation of our model as a system where each probability constraint can be viewed as a system component, each with its reliability in other words its tolerance measure. We develop a transformed programming model which may be specified mostly in terms of reliability measures. To form models of the form we redefine as variables for this model:

$$x_4 = F^{-1}(\alpha - \tfrac{1}{2})$$

$$x_5 = F^{-1}(\alpha - \tfrac{1}{2})$$

$$x_6 = F^{-1}(\alpha - \tfrac{1}{2})$$

$$x_7 = F^{-1}(\eta_2 - \tfrac{1}{2})$$

$$x_8 = F^{-1}(\eta_3 - \tfrac{1}{2})$$

System reliability $R = (u^*)^5$

u^* is defined in Equation [4.2.18]

$(1-R)$ = System unreliability

and we incorporate an additional restriction in order to have a measure of system reliability as a whole

$$x_4 \cdot x_5 \cdot x_6 \cdot x_7 \cdot x_8 \geq (u^*)^5 \qquad [4.2.18]$$

where u* is a preassigned value. And we also add five additional constraints as upper bounds for our variables x_4, x_5, x_6, x_7, and x_8. So we have

$$x_4 \leq 4.0 \qquad [4.2.19]$$

$$x_5 \leq 4.0 \qquad [4.2.20]$$

$$x_6 \leq 4.0 \qquad [4.2.21]$$

$$x_7 \leq 4.0 \qquad [4.2.22]$$

$$x_8 \leq 4.0 \qquad [4.2.23]$$

The upper bound value of 4.0 corresponds in the normal curve to a value of probability 99.999999%.

Then our last step is to associate a monotonic pay-off function for achieving the level R of system reliability. As we have seen before, one possible choice of utility function satisfying conditions given in Chapter 2 is

$$\text{Max } .05x_1 + .05x_2 + .05x_3 + \ln x_4 + \ln x_5 + \ln x_6 + \ln x_7 + \ln x_8 \qquad [4.2.24]$$

Therefore collecting the expressions [4.2.18], [4.2.19], [4.2.20], [4.2.21], [4.2.22], [4.2.23], and [4.2.24] and adding to our model [4.2.17] we got the following problem:

$$\text{Max } .05x_1 + .05x_2 + .05x_3 + \ln x_4 + \ln x_5 + \ln x_6 + \ln x_7 + \ln_8$$

subject to

$$g_1(x) \equiv 1300.+(.05-.15x_4)x_1 \geq 0$$

$$g_2(x) \equiv 1000.+(.05-.15x_5)x_2 \geq 0$$

$$g_3(x) \equiv 1000.+(.05-.15x_6)x_3 \geq 0$$

$$g_4(x) \equiv 7000. -x_1 \geq 0$$

$$g_5(x) \equiv 5500.+(.05-.15x_7)x_1 -x_2 \geq 0$$

$$g_6(x) \equiv .15x_8\sqrt{x_1^2+x_2^2}+9000.+.05x_1+.05x_2-x_3 \geq 0$$

$$g_7(x) \equiv x_4x_5x_6x_7x_8-(u^*)^5 \geq 0$$

$$g_8(x) \equiv 4.-x_4 \geq 0$$

$$g_9(x) \equiv 4.-x_5 \geq 0$$

$$g_{10}(x) \equiv 4.-x_6 \geq 0$$

$$g_{11}(x) \equiv 4.-x_7 \geq 0$$

$$g_{12}(x) \equiv 4.-x_8 \geq 0$$

The method for solving this specific problem using SUMT is given in Appendix B.

We have solved the following set of problems for different values of u* such as 1.29 (90%), 1.65 (95%) and 2.33 (99%). The results are given in Table 4.2.4.

In Figure 4.2.1 we observe a substitution curve between profit and system unreliability. We have a net profit of 999.43 with 10% of system unreliability if we want to decrease system unreliability to 5% we have to trade off some net profit for it, so now we only get 923.34. Further if we want to decrease system unreliability to 1% we increase our profit to 792.35.

If we define arc elasticity of profit-reliability as

$$\varepsilon_{pR} = - \frac{\frac{\partial p}{p}}{\frac{\partial R}{R}}$$

we find that the elasticity between the profit of 999.43 and 923.24 is 1.568 while the corresponding arc elasticity between the profit of 923.34 and 792.35 is 3.77.

We can find more points and give a more exact substitution curve than in Figure 4.2.1 that will show the range of choices available to the decision maker between differing levels of profit and system unreliability.

We can see in Figure 4.2.2 the changes in variable x_1 due to the change in system unreliability; this variable is very sensitive that is,only the second decimal figure changes for.

In Figure 4.2.3 we can observe the changes in variable x_2 due to difference in system unreliability e.g., x_2 decreases

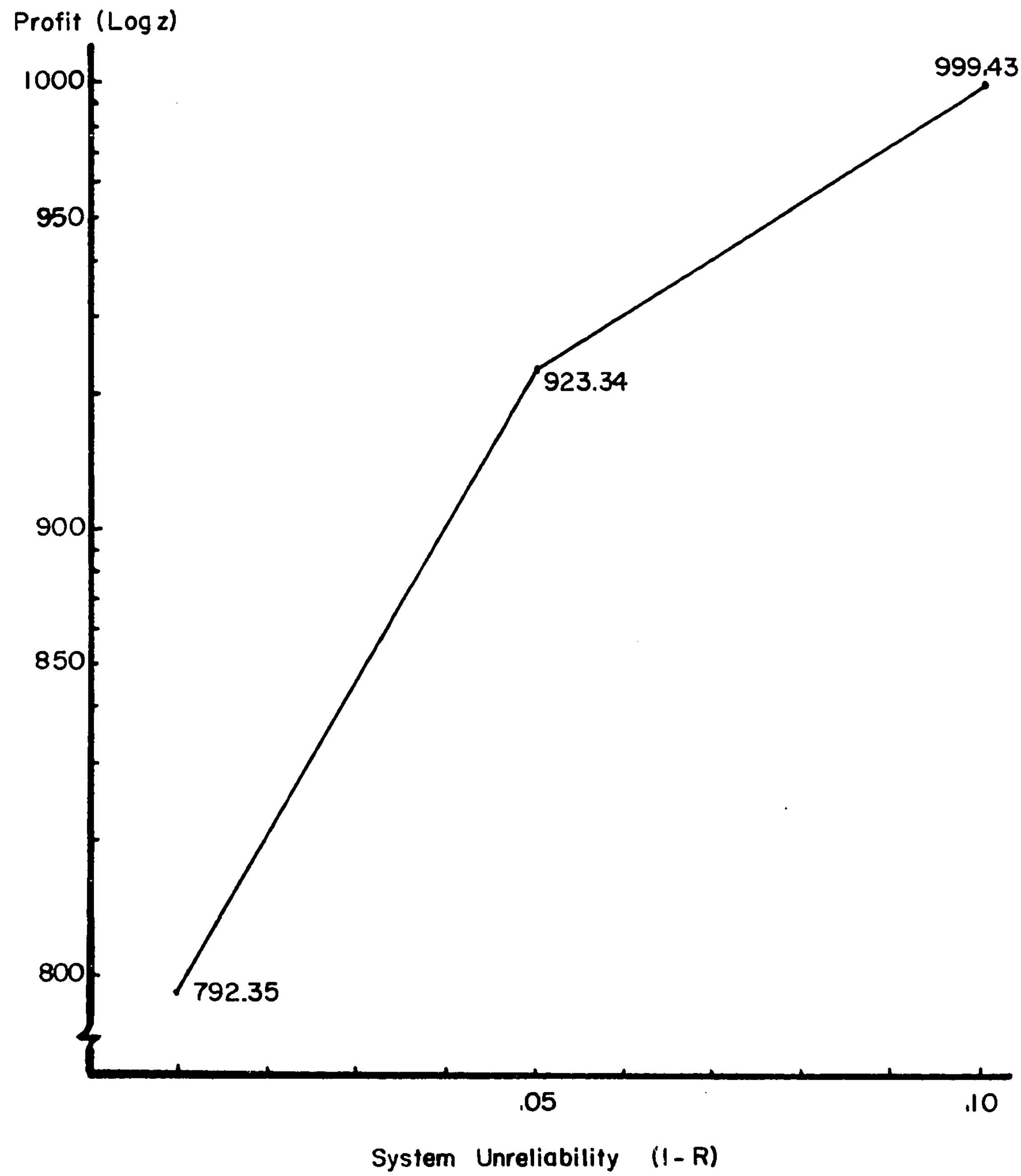

Figure 4.2.1 Substitution Curve between Profit and System Unreliability.

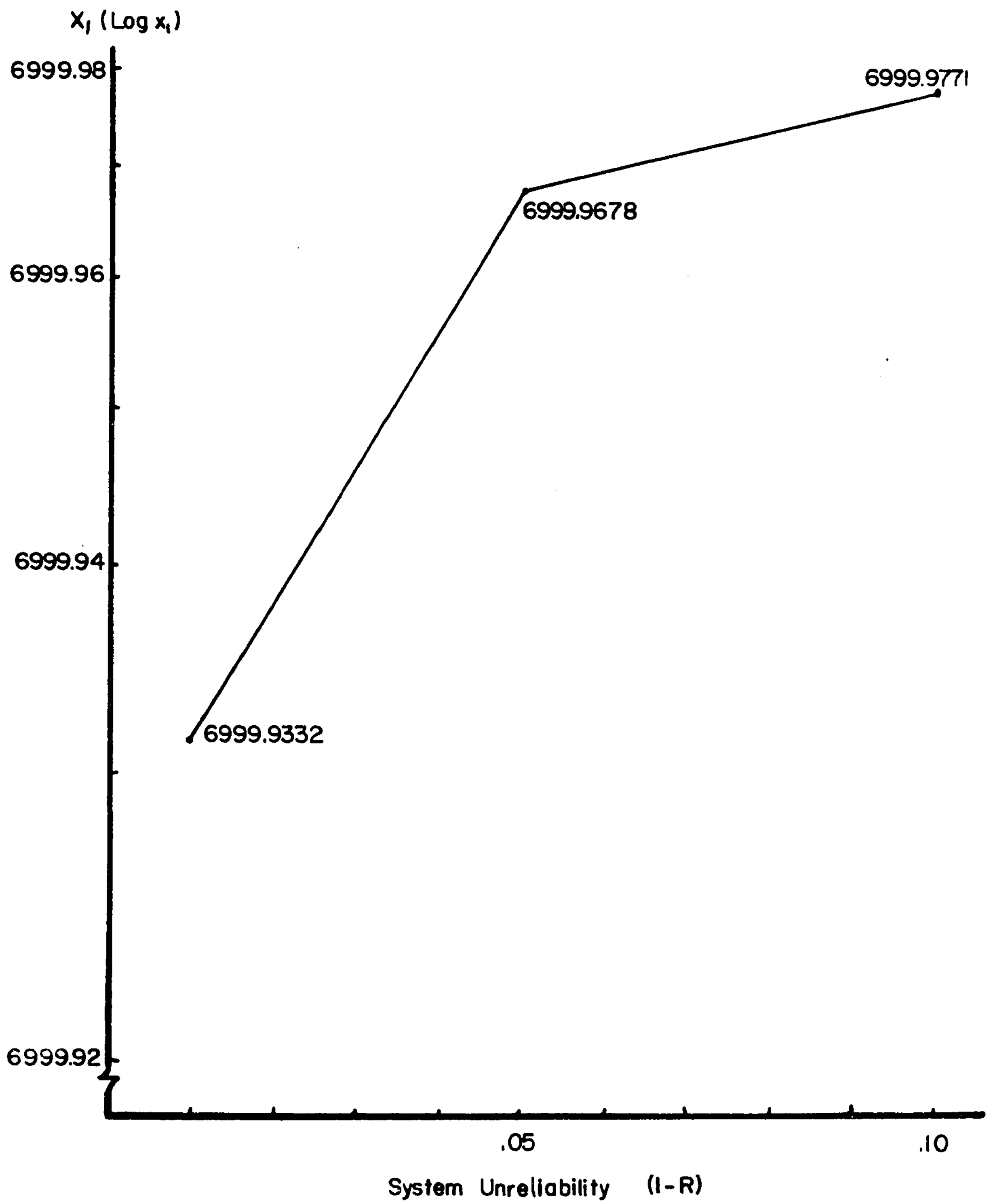

Figure 4.2.2. Curve between X_1 and System Unreliability.

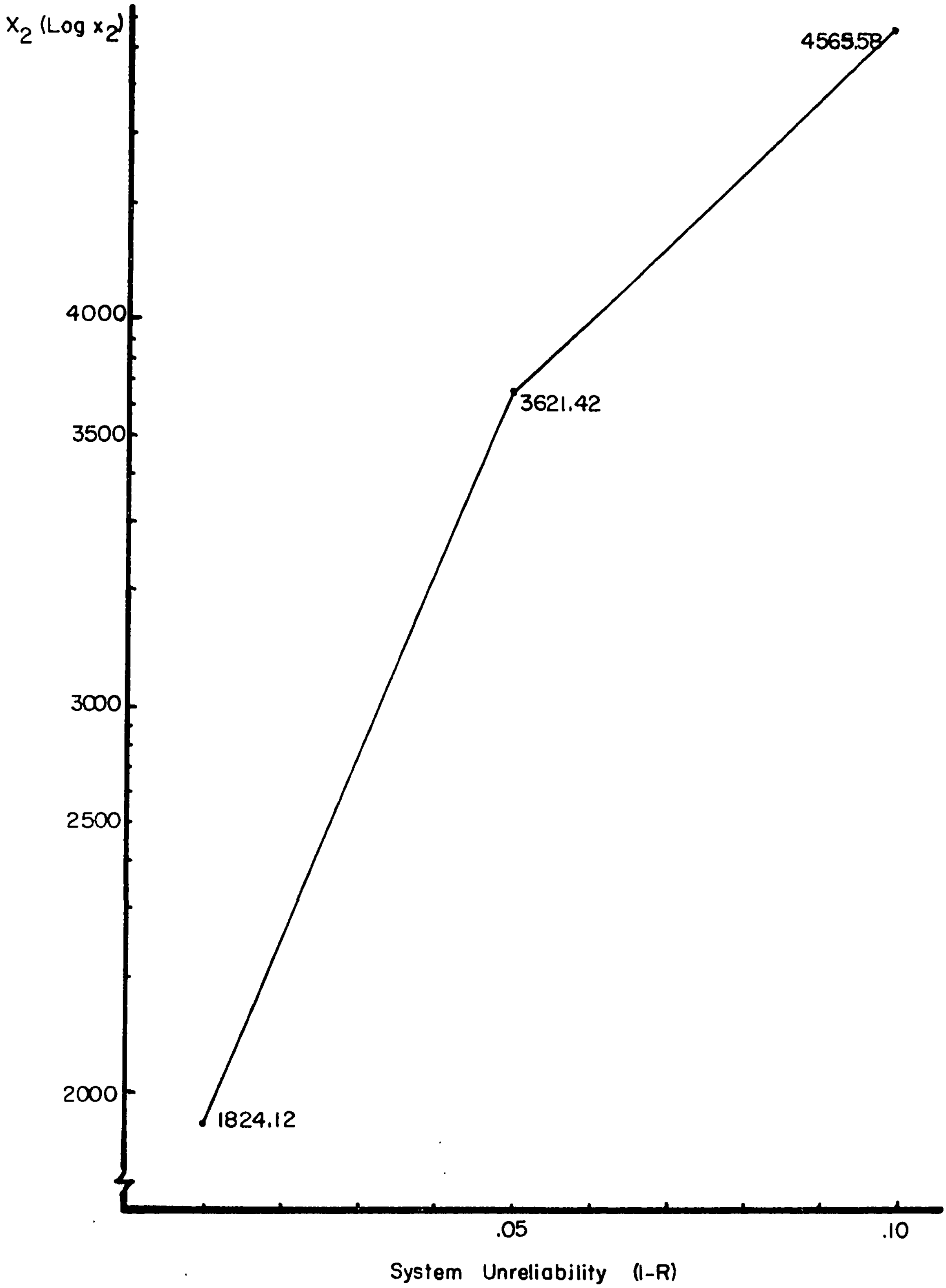

Figure 4.2.3. Curve between X_2 and Systems Unreliability.

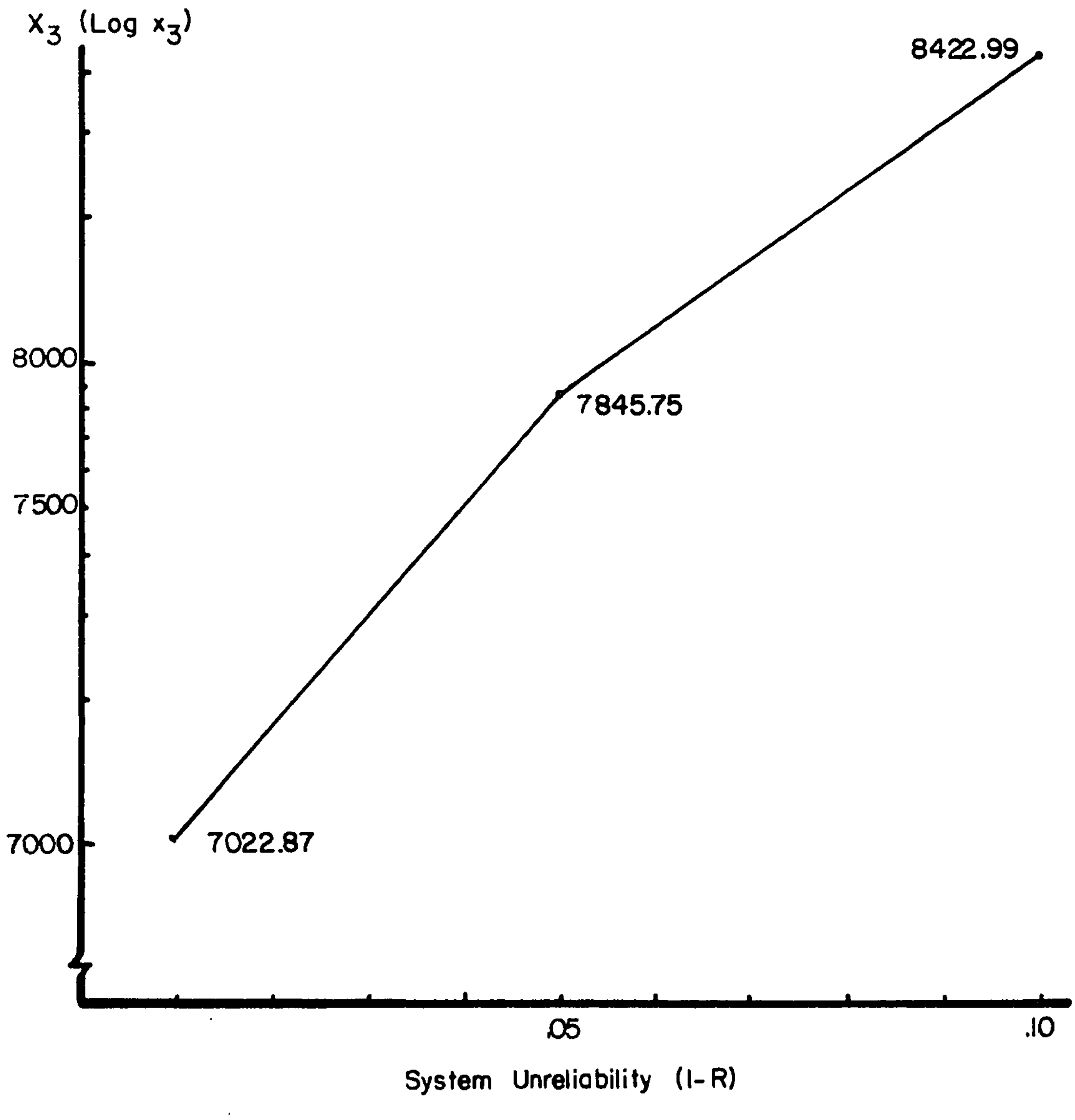

Figure 4.2.4. Curve between X_3 and Systems Unreliability

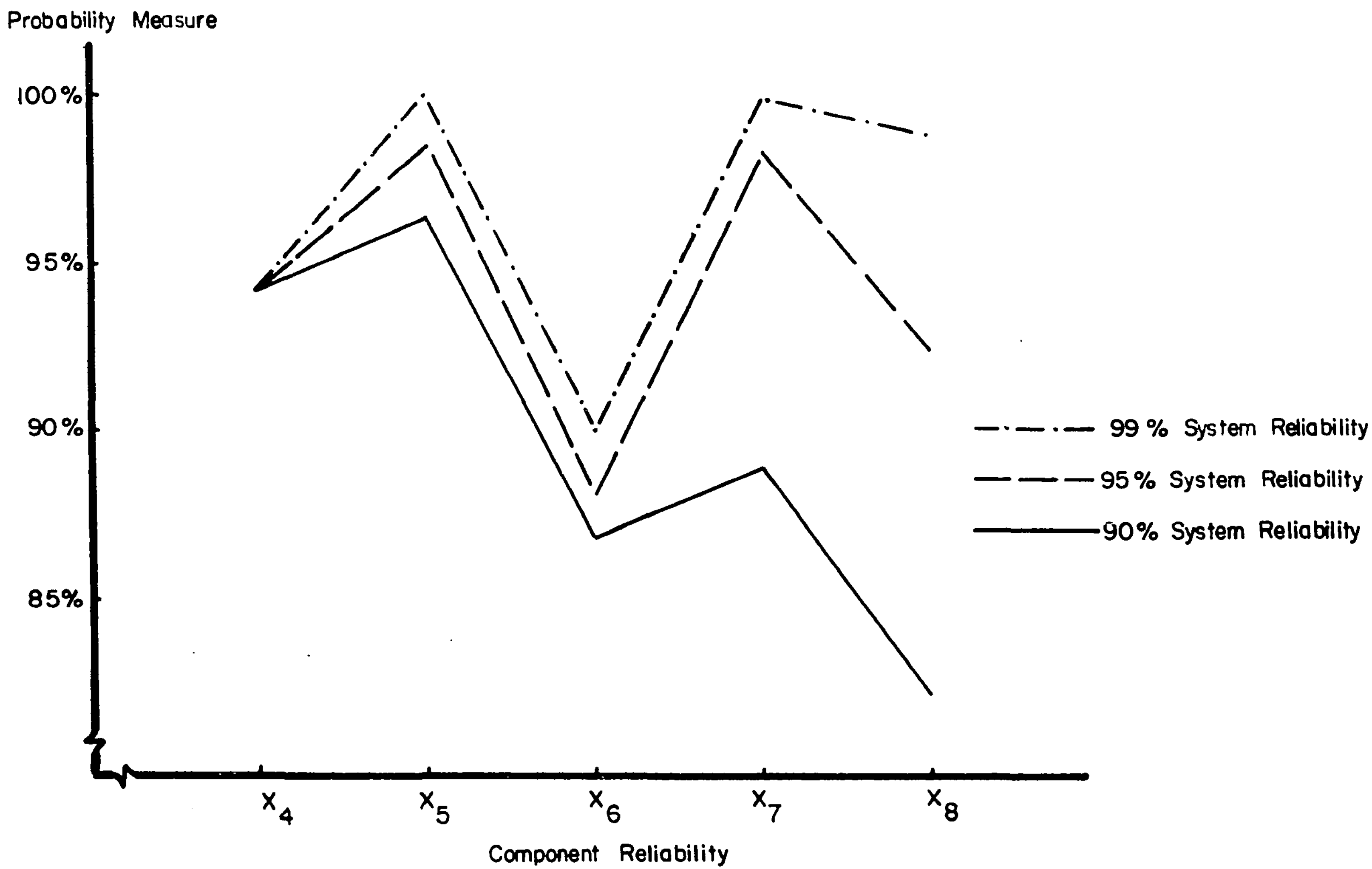

Figure 4.2.5. Curves of Component Reliability due to Changes in System Reliability.

from 4565.58 to 3621.42 due to a decrease of 5% of system unreliability, while for decreasing system unreliability to 1% (i.e. 4% less) we have to decrease x_2 to 1824.12, which is almost half the value x_2 has at 5%.

From Figure 4.2.4 we observe that a decrease of 5% of system unreliability that is from 10% to 5% produces a decrease of 577.60 in x_3 while x_3 varies 822.52 from a decrease from 5% to 1%.

Table 4.2.4. Optimal solutions for optimization models of reliability programming

	System Reliability $\geq(1.29)^5$ (90%)	System Reliability $\geq(1.65)^5$ (95%)	System Reliability $(2.33)^5$ (99%)
x_1	6999.9771	6999.9678	6999.9332
x_2	4565.5757	3621.4217	1824.1240
x_3	8422.9951	7845.3945	7022.8712
x_4	1.5714071 (94.18%)[a]	1.5714070 (94.18%)	1.5713972 (94.18%)
x_5	1.7935038 (96.33%)	2.1741886 (98.50%)	3.9879089 (100%)
x_6	1.1247981 (86.86%)	1.1830666 (88.10%)	1.2825744 (89.97%)
x_7	1.2232458 (88.88%)	2.1224378 (98.30%)	3.8341266 (99.99%)
x_8	0.92156265 (82.12%)	1.4258726 (92.36%)	2.2287115 (97.71%)

[a]Probability percent found in a normal distribution Table (14, pp. 127-134), rounded to two decimal figures.

Table 4.2.4 (Continued)

	System Reliability $\geq(1.29)^5$ (90%)	System Reliability $\geq(1.65)$ (95%)	System Reliability $(2.33)^5$ (99%)
z_1[b]	1000.7009	925.843232	796.57584
z_2[c]	999.42743	923.33920	792.34642

[b]The value of the objective function as specified in [4.2.24].

[c]The value of the objective function as specified in [4.2.17] that is without the reliability function.

From the above results we point out that variable x_2 is more sensitive in responses to system reliability changes while x_1 is less; x_3 has small changes compared to x_2; thus we can say that x_2 is the more crucial variable in our problem of reliability programming.

In Figure 4.2.5 we observe the changes in component reliability, in other words the probability assigned by the system itself in order to satisfy the stochastic constraints due to a change in the total system reliability.

Note that:

x_4 = the risk prescribed for period 1

x_5 = the risk prescribed for period 2

x_6 = the risk prescribed for period 3

x_7 = the risk level for the capital constraint in period 2

x_8 = the risk level for the capital constraint in period 3

For our first probabilistic constraint we have a common value of 94.18% that is x_4 is the same for the three values of system reliability. Greater changes occur with 90% system reliability than with 95% and 99% levels. We observe that in the three cases the changes in the components move in the same direction that is, increase or decrease in every component for difference in system reliability. The largest difference in component reliability for difference in system reliability is x_8 which indicates that our last constrained restriction is more sensitive to changes in the total system reliability.

Table 4.2.5. Lagrange multipliers associated with optimization models of reliability programming

Constraints	System Reliability $\geq(1.29)^5$ (90%)	System Reliability $\geq(1.65)^5$ (95%)	System Reliability $(2.33)^5$ (99%)
$g_1(x)$	0.026795	0.039712	0.042705
$g_2(x)$	0.038744	0.06146	0.054547
$g_3(x)$	0.031901	0.047656	0.053781
$g_4(x)$	0.039046	0.031859	0.019863
$g_5(x)$	0.051422	0.043254	0.014314
$g_6(x)$	0.055033	0.054674	0.028551
$g_7(x)$	11.773290	4.944558	1.175800
$g_8(x)$	3.29×10^{-6}	5.60×10^{-6}	1.50×10^{-5}
$g_9(x)$	3.98×10^{-6}	9.92×10^{-6}	0.606001
$g_{10}(x)$	2.35×10^{-6}	4.17×10^{-6}	1.19×10^{-5}
$g_{11}(x)$	2.52×10^{-6}	9.38×10^{-6}	3.22×10^{-3}
$g_{12}(x)$	2.05×10^{-6}	4.99×10^{-6}	2.82×10^{-5}

In Table 4.2.5 we indicate the Lagrange multiplier for the three system reliability problems. The most binding constraint is the measure of system reliability (i.e., $g_7(x)$).

$g_1(x)$

4.2.2. Sample distribution approach

We have assumed that $(p_i-p_{i-1})/p_{i-1}$ is approximately normally distributed with mean $\mu=.05$ and variance $\sigma^2=.0225$. Note however that μ and σ^2 are population parameters of the normal parent $N(\mu,\sigma^2)$ and these are not generally observable, i.e., these are unknown constants which are to be estimated from sample observations, unless of course the random variation in $(p_i-p_{i-1})/p_{i-1}$ around μ are generated by a controlled experiment. The later case may arise for example when in a simulation study random numbers are drawn from a normal table with preassigned (i.e. known) mean μ and variance σ^2. However in most economic models and other fields of application the parameters μ, σ are unknown and have to be estimated from sample non-experimental observations. Now usually if the sample observations on each $(p_i-p_{i-1})/p_{i-1}$ are available and these can be seasonably assumed to be random drawings from a normal parent $N(\mu,\sigma^2)$, then the sample mean $\overline{X}$ provides a good estimate of μ; similarly in this case a good estimate $\hat{\sigma}^2$ of σ^2 may be easily defined. We have to note that if the sample mean $\overline{x}$ is used as unbiased estimate

($E\bar{x}=\mu$) of the parameter μ, we should refer to the distribution of the sample mean rather than the population distribution. But since $\bar{x}$ is normally distributed with expectation μ and variance σ^2/T, our problem [4.2.17] will be:

$$\text{Max } .05x_1 + .05x_2 + .05x_3$$

subject to

$$\begin{array}{llllll}
g_1(x) \equiv & 1300+(.05-\frac{.15}{\sqrt{T}}\psi)x_1 & & & & \geq 0 \\
g_2(x) \equiv & 1000 & +(.05-\frac{.15}{\sqrt{T}}\ )x_2 & & & \geq 0 \\
g_3(x) \equiv & 1000 & & +(.05-\frac{.15}{\sqrt{T}}\psi)x_3 & & \geq 0 \\
g_4(x) \equiv & 7000 & -x_1 & & & \geq 0 \\
g_5(x) \equiv & 5500+(.05-\frac{.15}{\sqrt{T}}\Psi)x_1 & & -x_2 & & \geq 0 \\
g_6(x) \equiv \frac{-.15}{\sqrt{T}}\phi\sqrt{x_1^2+x_2^2} & +9000 & +.05x_1 & +.05x_2 & -x_3 & \geq 0
\end{array}$$

In Table 4.2.6 we show different solutions of the above problem for sample size of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144 units. We observe the increase in profit as T increase, we have that with 4 units we got a profit of 934.32 while for 144 units we got 1091.42 of profit.

From Figure 4.2.6 we can see that the curve of profit due to changes in the sample size increases rapidly in profit at the beginning. Then it has a slow variation that implies a diminishing return of profit due to an increase in the sample size.

One possible way to find an optimal sample size will be to draw a tangent line to the profit-sample size curve which is parallel to net line

$$z = a+b\sqrt{n}$$

where

z = profit

$\sqrt{n}$ = square root of the sample size

a = intercept

b = slope

That point will give us how much profitwe will get with that sample size.

Let us have an example with $a=0$, $b=10$ so our line is $z=10\sqrt{n}$, the optimal point may be $\sqrt{n}=6$ and $z=1058.84$. Many other points could be found if we knew the parameters a, and b.

In the case the sample size increases to infinite we observe that the term $.15\sqrt{T}$ goes to zero and our nonlinear programming becomes the following linear programming problem.

$$\text{Max } .05x_1 + .05x_2 + .05x_3$$

Table 4.2.6. Profit variation due to sample size variation[a]

$\sqrt{T}$	Sample T	z	x_1	x_2	x_3
1	1	756.74	6581.35	3528.65	5024.71
2	4	934.52	6999.78	4626.58	7063.95
3	9	995.55	6999.72	5034.26	7877.03
4	16	1026.65	6999.79	5238.18	8300.29
5	25	1045.99	6999.72	5360.44	8559.76
6	36	1058.84	6999.69	5441.95	8735.16
7	49	1068.08	6999.70	5500.50	8861.65
8	64	1075.07	6999.897	5544.08	8957.39
9	81	1080.50	6999.876	5578.04	9032.07
10	100	1084.86	6999.86	5605.21	9092.07
11	121	1088.43	6999.856	5627.44	9141.32
12	144	1091.42	6999.85	5645.97	9182.47
∞	∞	1124.63	7000.0	5850.0	9462.50

[a]Decimal values are rounded.

subject to

$$7000 - x_1 \geq 0$$

$$5500 + .05x_1 - x_2 \geq 0$$

$$9000 + .05x_1 + .05x_2 - x_3 \geq 0$$

$$x_1, x_2, x_3 \geq 0$$

Solving this problem using MPS (Mathematical Programming System) (69, 70), we get an optimal solution as x_1=7000., x_2=5850., x_3=9642.50 with a value of 1124.63.

In Table 4.2.7 we indicate the Lagrange multipliers associated with the sample size variation problems. The most binding constraints are the last three constraints.

We can develop other models where it could include costs of sampling in the objective function.

Another way we could develop this model further is by considering sample distribution problems indicated in (70).

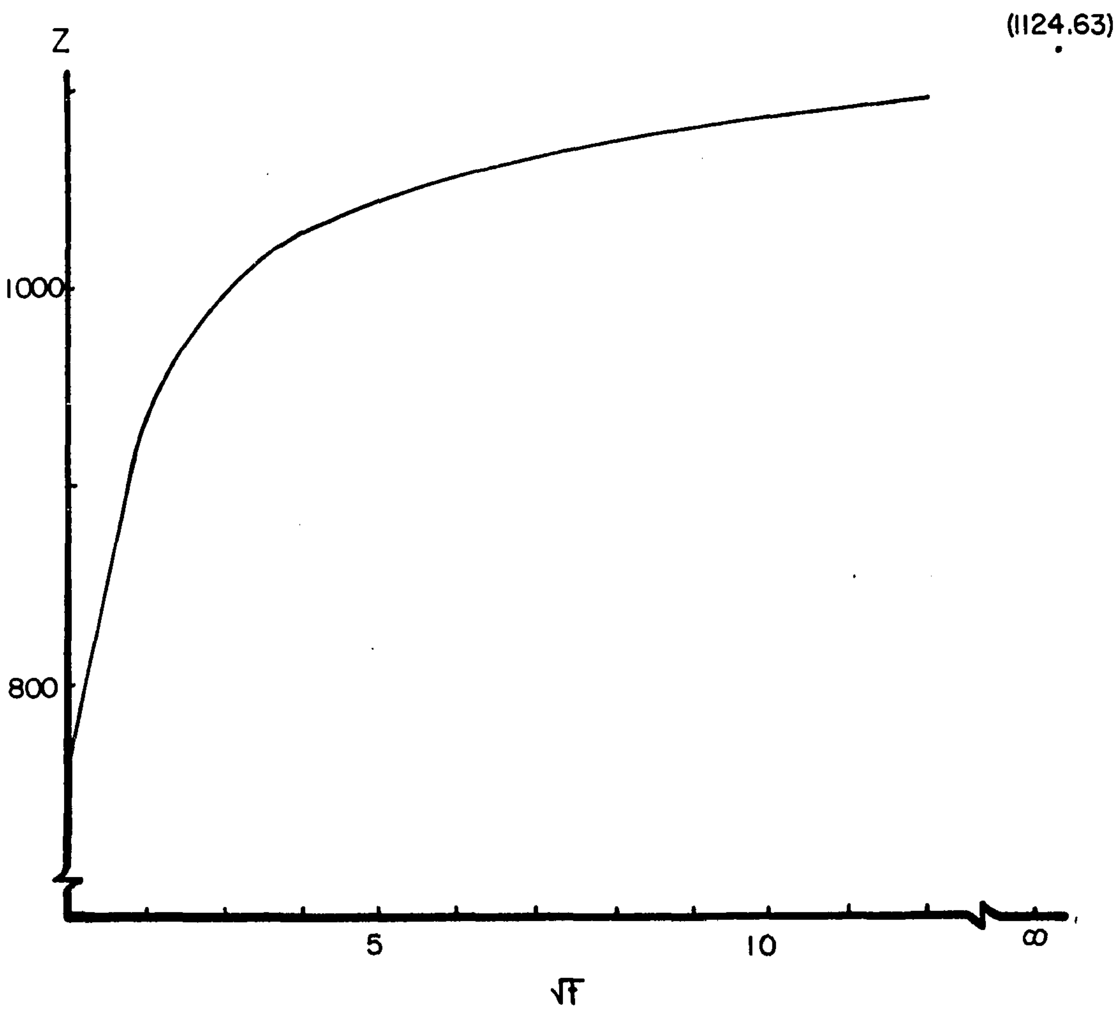

Figure 4.2.6. Variation of Profit due to Change in Sample Size

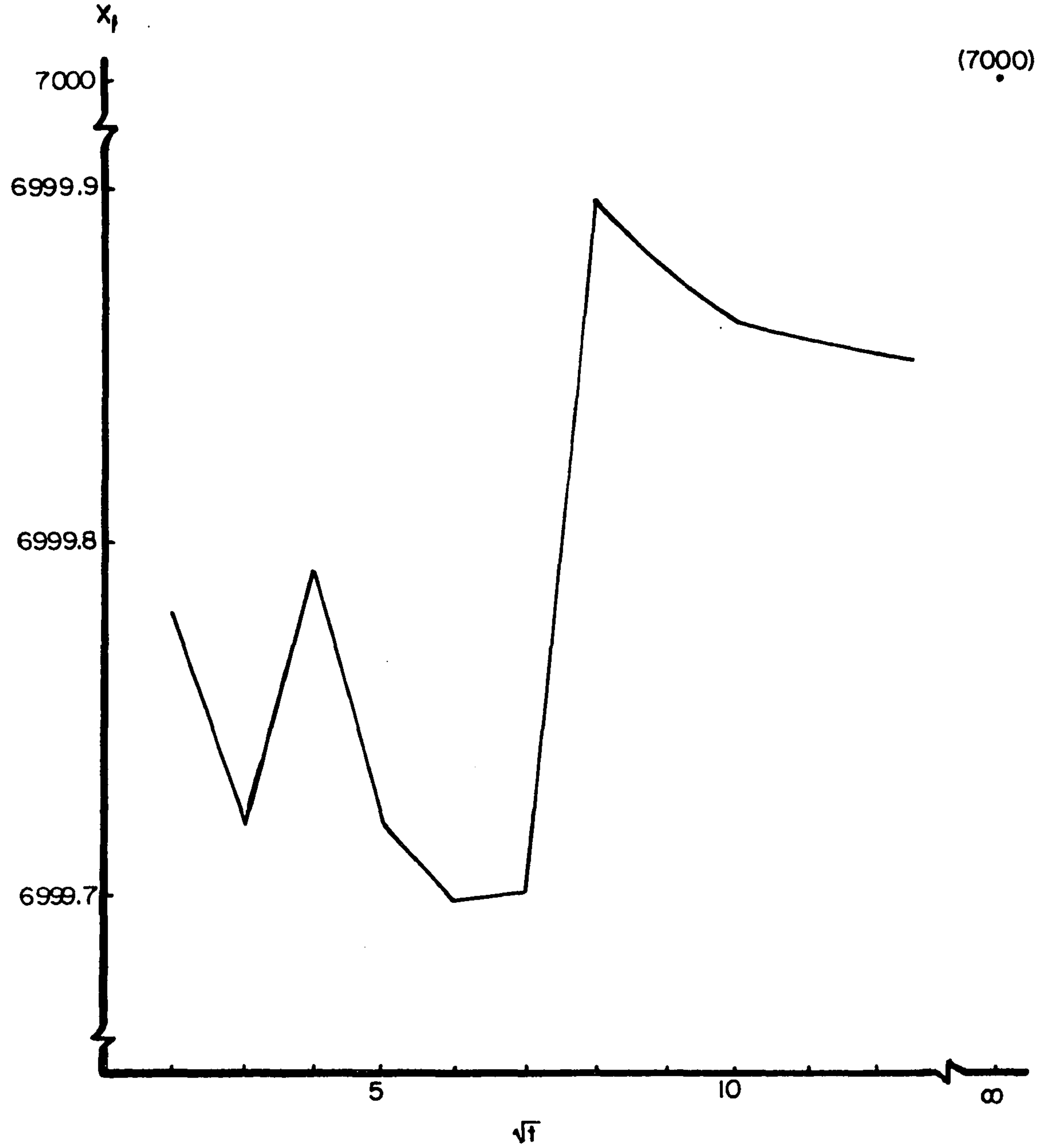

Figure 4.2.7. Variation of X_1 due to Change in Sample Size

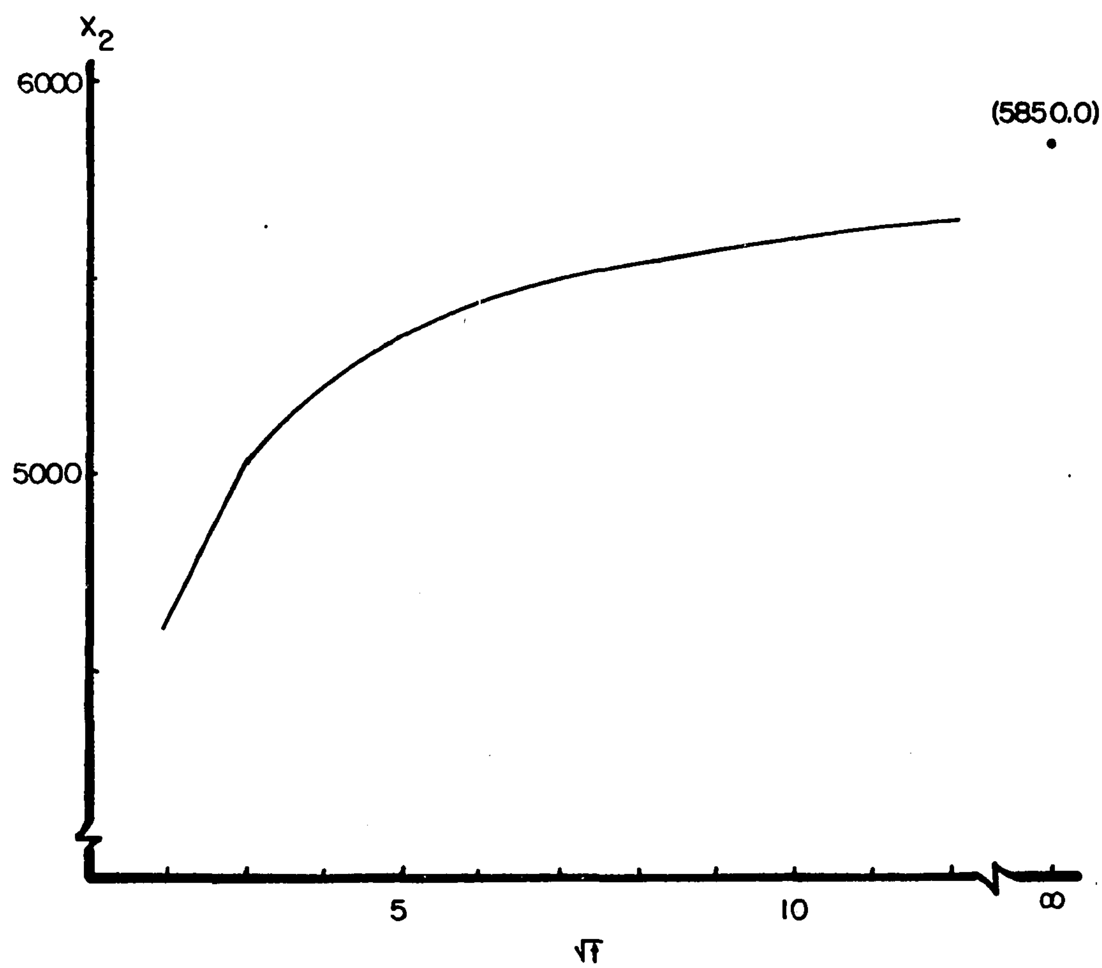

Figure 4.2.8. Variation of X_2 due to Change in Sample Size.

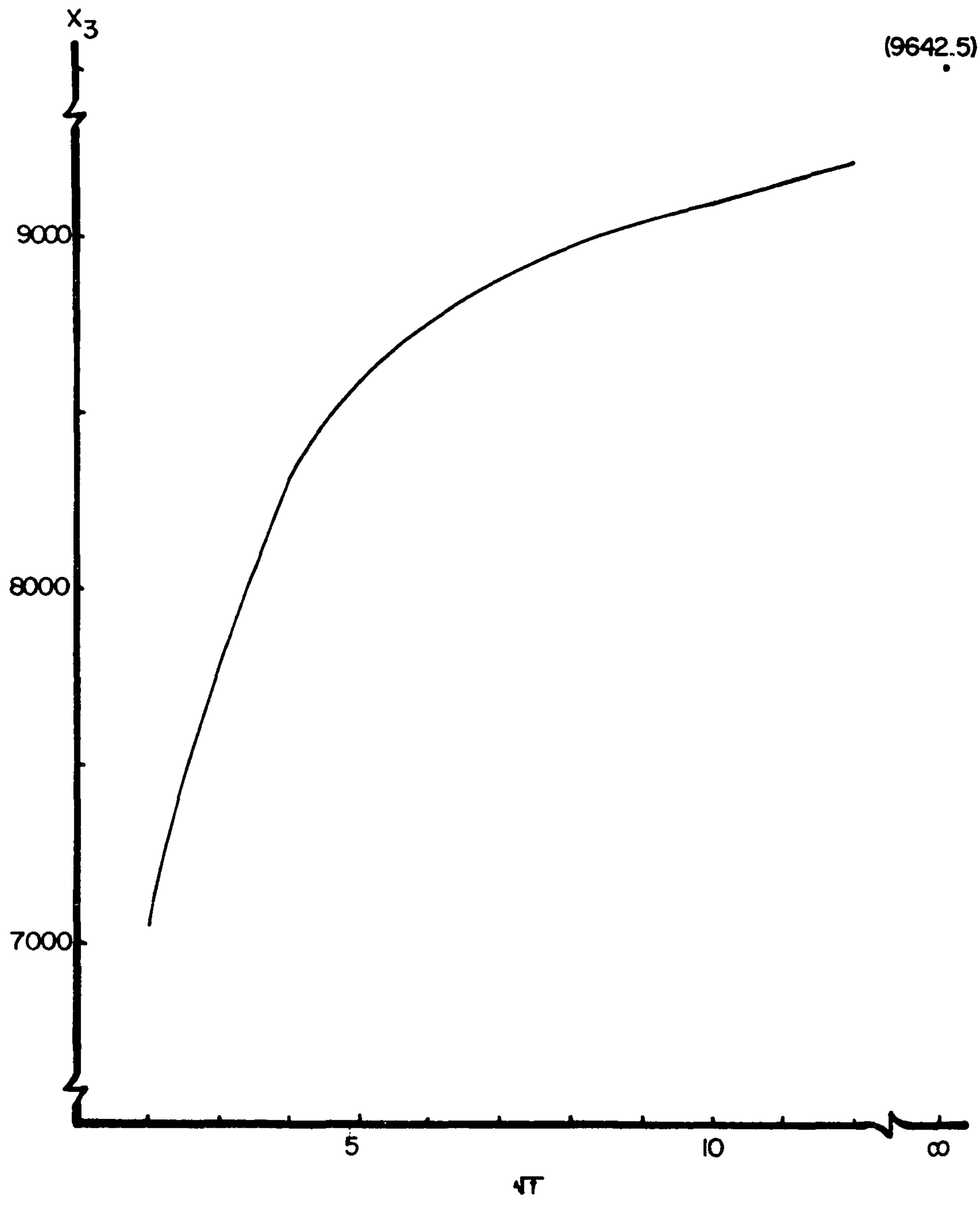

Figure 4.2.9. Variation of X_3 due to Change in Sample Size

Table 4.2.7. Lagrange multipliers associated with the sample size variation problems

Constraint	T=1	T=4	T=9	T=16	T=25	T=36
$g_1(x)$	0.061133	2.86×10^{-9}	2.79×10^{-9}	1.31×10^{-9}	2.09×10^{-9}	2.38×10^{-9}
$g_2(x)$	2.21×10^{-8}	4.06×10^{-9}	4.59×10^{-9}	2.20×10^{-9}	3.54×10^{-9}	4.02×10^{-9}
$g_3(x)$	3.49×10^{-5}	7.67×10^{-9}	5.81×10^{-9}	2.38×10^{-9}	3.53×10^{-9}	3.80×10^{-9}
$g_4(x)$	1.16×10^{-8}	0.035103	0.040950	0.044974	0.046905	0.047968
$g_5(x)$	0.035969	0.049626	0.046335	0.048327	0.048909	0.049490
$g_6(x)$	0.046159	0.051142	0.049572	0.050481	0.050284	0.049591

Table 4.2.7 (Continued)

Constraint	T=49	T=64	T=81	T=100	T=121	T=144	T=∞
$g_1(x)$	2.33×10^{-9}	2.62×10^{-10}	3.66×10^{-10}	4.33×10^{-10}	4.79×10^{-10}	5.12×10^{-10}	-
$g_2(x)$	3.92×10^{-9}	4.39×10^{-10}	6.14×10^{-10}	7.24×10^{-10}	7.99×10^{-10}	8.55×10^{-10}	-
$g_3(x)$	3.59×10^{-9}	3.92×10^{-10}	5.37×10^{-10}	6.24×10^{-10}	6.81×10^{-10}	7.20×10^{-10}	-
$g_4(x)$	0.049226	0.050290	0.050679	0.051076	0.051444	0.051604	0.05512
$g_5(x)$	0.049789	0.049784	0.049806	0.049970	0.050234	0.050328	0.05250
$g_6(x)$	0.050149	0.050454	0.050334	0.050296	0.050299	0.050180	0.05000

5. APPLIED GEOMETRIC PROGRAMMING

5.1. A Simple Stochastic Production Model: an Illustration

An entrepreneur transforms inputs into outputs, subject to the technical rules specified by his production function.

The entrepreneur's production function gives mathematical expression to the relationship between the quantities of inputs he employs and the quantity of output he produces. A specific production function may be given by a single point, a single continuous or discontinuous function, or a system of equations (63).

We limit here to a production function given by a single continuous function with continuous first- and second-order partial derivates. The analysis is first developed for the simple case in which three inputs are combined for the production of a single output and then extended to stochastic case.

So we have three inputs called, x_1, labor; x_2, raw materials and x_3 capital, the cost of production c is given by the linear equation

$$c = .14x_1 + .04x_2 + .06x_3$$

the production function is given by:

$$y = 100\, x_1^{1.6} x_2^{.5} x_3^{.7}$$

Note that this production function assumes increasing returns to scale, the following

$$y \geq 1000$$

and labor to capital ratio must be

$$\frac{x_1}{x_3} \leq 10; \ \frac{x_1}{x_3} \geq 5$$

then we will have the following programming problem

$$\text{Min } .14x_1 + .04x_2 + .06x_3$$

subject to

$$100x_1^{1.6}x_2^{.5}x_3^{.7} \geq 1000$$

$$\frac{x_1}{x_3} \leq 10$$

$$\frac{x_1}{x_3} \geq 5$$

The geometric programming formulation will be

$$\text{Min } .14x_1 + .04x_2 + .06x_3$$

subject to

$$10x_1^{-1.6}x_2^{-.5}x_3^{-.7} \leq 1$$

$$.1x_1x_3^{-1} \leq 1$$

$$5x_1^{-1}x_3 \leq 1$$

The degree of difficulty is two, because we have six terms and three variables, the dual will be

$$\text{Max}\left(\frac{.14}{\delta_1}\right)^{\delta_1}\left(\frac{.04}{\delta_2}\right)^{\delta_2}\left(\frac{.06}{\delta_3}\right)^{\delta_3}\left(\frac{10}{\delta_4}\right)^{\delta_4}\left(\frac{.1}{\delta_5}\right)^{\delta_5}\left(\frac{5}{\delta_6}\right)^{\delta_6}$$

subject to

$$
\begin{array}{llllllll}
\delta_1 + \delta_2 + \delta_3 & & & & & & = 1 \\
\delta_1 & & -1.6\delta_4 & + & \delta_5 & - \delta_6 & = 0 \\
& \delta_2 & -.5\delta & & & & = 0 \\
& & \delta_3 - \delta_4 & & \delta_5 & + \delta_6 & = 0
\end{array}
$$

$$\delta_1, \delta_2, \delta_3 \geq 0$$

Solving the problem we have that:

$g_0(x)=.6516$, $x_1=3.5213$, $x_2=2.9089$, $x_3=0.7043$

and the dual variables are:

$v(\delta)=.6516$, $\delta_1=0.75658$, $\delta_2=0.17857$, $\delta_3=0.06485$

$\delta_4=0.35714$, $\delta_5=5.347\times10^{-8}$, $\delta_6=0.18515$

as we have two degrees of difficulty we do not have any economical interpretation of the dual variables.

As an easy way to deal with a stochastic case we can assume that production has to match sales, which is a random variable with normal distribution with mean equal to 1000 units and variance equal to 100, so our model now is

$$\text{Min } .14x_1 + .04x_2 + .06x_3 \qquad [5.1.1]$$

subject to

$$\text{Prob}(y \geq y^*) \geq u \qquad [5.1.2]$$

$$y = 100x_1^{1.6}x_2^{.5}x_3^{.7}$$

$$\frac{x_1}{x_3} \leq 10$$

$$\frac{x_1}{x_3} \geq 5$$

where y^* is a random variable with a normal distribution with $\bar{y}=1000$ and $\sigma^2=100$. The constraint [5.1.2] will be

$$\text{Prob}(y^* \leq 100x_1^{1.6}x_2^{.5}x_3^{.7}) \geq u$$

$$\text{Prob}\left(\frac{y^*-\bar{y}}{\sigma} \leq \frac{100x_1^{1.6}x_2^{.5}x_3^{.7}-\bar{y}}{\sigma}\right) \geq u$$

now if we define

$$I(x) = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

we have

$$I\left(\frac{100x_1^{1.6}x_2^{.5}x_3^{.7}-\bar{y}}{\sigma}\right) \geq u$$

$$\frac{100x_1^{1.6}x_2^{.5}x_3^{.7}-\bar{y}}{\sigma} \geq I^{-1}(u)$$

Let us make $I^{-1}(u) = x_4$ then

$$100x_1^{1.6}x_2^{.5}x_3^{.7} \geq \bar{y}+\sigma I^{-1}(u)$$

$$100x_1^{1.6}x_2^{.5}x_3^{.7} \geq 1000 + 10x_4$$

Then we will have the following geometric problem

$$\text{Min } .14x_1 + .04x_2 + .06x_3 + x_4$$

subject to

$$100x_4^{-1} - 10x_1^{1.6}x_2^{.5}x_3^{.7} \le 1$$

$$.1x_1x_3 \le 1$$

$$5x_1^{-1}x_3 \le 1$$

$$1.65x_4^{-1} \le 1$$

we note that we have added a penalty function x_4 to the objective function and added a lower bound to x_4 (i.e. $x_4 \ge a^*$) in the present case we assume that the probability level u must be greater than 90%, 97%, 99%. We note that the first constraint is not a posynomial (positive polynomial) so we are dealing with the case of a generalized polynomial programming (138, 16). The problem has four degrees of difficulty and the results are given in Table 5.1.1 and the dual results are given in Table 5.1.2.

Table 5.1.1. Primal results

Variable	a*=1.65 90%	a*=1.88 97%	a*=2.33 99%
x_1	2.9272	2.7915	2.5814
x_2	2.4181	2.3061	2.1324
x_3	0.5854	0.5583	0.5163
x_4	1.65	1.88	2.33
z	.5417	.5166	.4377

Table 5.1.2. Dual results

Variable	a*=1.65 90%	a*=1.88 97%	a*=2.33 99%
δ_1	0.1869	0.1631	0.1287
δ_2	0.0441	0.0385	0.0304
δ_3	0.0160	0.0139	0.0103
δ_4	0.7529	0.7845	0.8299
δ_5	0.0897	0.0785	0.0622
δ_6	0.0883	0.0769	0.0608
δ_7	4.24×10^{-8}	4.50×10^{-8}	4.43×10^{-8}
δ_8	0.0458	0.0399	0.0315
δ_9	0.6631	0.7060	0.7677

5.2. An Aggregative Model of Economic Growth: a Second Illustration

Hirofumi Uzawa (134), discussed the problem of optimum fiscal policy in terms of the technique of optimum economic growth. He pointed out that his model is a simple extension of the aggregative growth model of the type introduced by Solow (124), Swan (127) and Tobin (133). We use Uzawa's model, in order to show an application of geometric programming, with some simplifying assumptions.

The Uzawa model consists of private and public sectors, both employing labor and private capital to produce goods and services. Private goods may be either consumed or accumulated as capital, while public goods are all consumed. The public sector raises revenues by levying income taxes. The private sector decides how much is to be consumed and invested and how to allocate portfolio balances between real capital and money. These decisions are based upon certain behavioristic assumptions and are made in a perfectly competitive institutional setting. Uzawa had shown that by a proper choice of dynamic fiscal policy, which consists of income tax rates and growth rates of money supply through time, it is possible to achieve an optimal growth path corresponding to certain forms of social utility function properly discounted.

We consider an economic system composed of public and private sectors. The private sector comprises business firms and households; business firms employ labor and either own or rented capital, while households receive wages for the labor they provide and interest and dividends for the capital they rent to business firms. The output produced in the private sector is assumed to be composed of homogeneous quantities so that any portion of it may be either instantaneously consumed or accumulated as part of the capital stock. The public sector will provide the private sector with different goods and services than those it produces. The goods and services produced in the public sector are assumed to be measurable and distributed uniformly to the private sector without cost. The public sector raises revenues through income taxes and increases in the money supply to pay wages and rentals for the private means of production it employs. It is required to employ labor and capital in such a way that total expenditure is minimized for any level of public goods produced. The public sector has two means of raising revenues, they are taxation and printing money, but is assumed to be able to control only the rate of income tax and the rate of increase in money supply. Capital accumulation takes place only in the private sector and public goods are not accumulated. Uzawa has analyzed the problem of optimum investment in public capital in which all public goods are regarded as social capital to increase productivity of labor and as private capital in the private sector.

The aggregation output at moment t, $Y_c(t)$ in the private sector is assumed to depend only on the amounts of capital $K_c(t)$ and labor, $L_c(t)$, employed, so we have a private sector's productive function F_c then

$$Y_c(t) = F_c(K_c(t),\ L_c(t)) \qquad [5.2.1]$$

The production processes are assumed to be subject to constant return to scale and to a positive diminishing marginal rate of substitution between capital and labor. Perfect competition prevails in the private sector, so that the real wage, w(t), and the real rental rate, r(t), are equated to the marginal products of labor and capital, respectively

$$w(t) = \frac{\partial F_c}{\partial L_c} \qquad r(t) = \frac{\partial F_c}{\partial K_c}$$

The aggregative output in the public sector at time t, is $Y_v(t)$ and it depends on capital $K_v(t)$ and labor $L_v(t)$ which it employs in such a combination that the total cost in terms of market prices is always minimized. So we have

$$Y_v(t) = F_v(K_v(t), L_v(t)) \qquad [5.2.2]$$

the public sector's production function, F_v, is assumed to be homogeneous of order one and strictly quasi-concave, with positive marginal products everywhere.

We know by the marginal theory that, the cost in the public sector is minimized when the marginal rate of substitution between labor and capital is equated to the wage-rentals ratio $w(t)/r(t)$, at time t.

$$\frac{\partial F_v/\partial L_v}{\partial F_v/\partial K_v} = \frac{w(t)}{r(t)}$$

the real gross national income $Y(t)$, is given by

$$Y(t) = r(t)K(t) + w(t)L(t) \qquad [5.2.3]$$

Since the quantities of capital and labor available are $K(t)$ and $L(t)$, we have

$$K(t) = K_c(t) + K_v(t) \qquad [5.2.4]$$

$$L(t) = L_c(t) + L_v(t) \qquad [5.2.5]$$

The output of private goods, $Y_c(t)$ is divided between consumption, $C(t)$ and investment, $Z(t)$ so:

$$Y_c(t) = C(t) + Z(t) \qquad [5.2.6]$$

The accumulation of capital is described by

$$\dot{K}(t) = Z(t) - \mu K(t) \qquad [5.2.7]$$

where μ is the rate of depreciation, while the growth rate of labor, n, is assumed to be exogeneously given:

$$\dot{L}(t) = nL(t) \qquad [5.2.8]$$

The desired level of investment, on the other hand, is determined by the Keynesian principle of marginal efficiency of investment, as mathematically formulated by Uzawa in (135). Business firms in the private sector try to increase the level of investment to that at which the supply price of capital is equated with the demand price (defined as the discounted sum of the expected return). In general, the desired level of investment, $Z(t)$ is determined by the current rate of return, $r(t)$, the money price, $p(t)$, the rate of interest $p(t)$, and the stock of capital, $K(t)$.

As Uzawa pointed out public goods are regarded as consumption goods, while private goods can either be consumed instantaneously or accumulated as capital. The utility function of representative members of the society depends upon the amount of private goods to be consumed and upon the average quantity of public goods available at each moment, for the sake of simplicity. Public goods are assumed to be distributed equally among the members of the society. If we defined $u(c,x)$ as a utility function where c and x stand respectively for the quantities of per capita consumption of private and public goods. He assumed that the social welfare function is represented as the discounted sum of instantaneous utilities through time; then we have

$$\int_0^{\infty} u(c,x)e^{-\delta t}\, dt \qquad [5.2.9]$$

where δ is the rate by which future utilities are compared with the present utilities with a proper modification when the population is not stationary.

Uzawa reformulates the problem of optimum fiscal policy, as this, he supposes that the public sector can determine or influence not only the fiscal policy but also the allocations of capital and labor between sectors and the division of private goods between consumption and investment. The public sector then seeks that feasible time-path of factor and output allocations at which the utility functional (5.2.9) is maximized. This time path, if it exists, will be called optimum path of economic growth.

So we have the following programming problem

$$\text{Max} \int_0^{\infty} u\left[\frac{C(t)}{L(t)}, \frac{X(t)}{L(t)}\right] e^{-\delta t}\, dt \qquad [5.2.10]$$

subject to

$$C(t) + Z(t) \leq F_c(K_c(t), L_c(t))$$
$$X(t) \leq F_v(K_v(t), L_v(t))$$
$$K_c(t) + K_v(t) \leq K(t)$$
$$L_c(t) + L_v(t) \leq L(t)$$
$$\dot{K}(t) = \quad (t) - \mu K(t)$$
$$\dot{L}(t) = nL(t)$$

when all variables are nonnegative and the initial capital K(0) and labor L(0) are given.

If we omit the time suffix for the sake of simplicity and we use small letters to indicate the quantities per capita, the programming problem of optimum economic growth is reduced to the following

$$\text{Max} \int_0^{\infty} u(c,x) e^{-\delta t}\, dt \qquad [5.2.11]$$

subject to

$$c+z \leq f_c(k_c) l_c \qquad [5.2.12]$$
$$x \leq f_v(k_v) l_v \qquad [5.2.13]$$
$$k_c l_c + k_v l_v \leq k \qquad [5.2.14]$$
$$l_c + l_v \leq 1 \qquad [5.2.15]$$
$$\dot{k} \geq z - (n+\mu)k \qquad [5.2.16]$$

We will try to solve the above programming problem, with some simplifying assumptions, first we assume a static problem and the change of capital stock is constant and known:

$$\frac{\dot{k}}{k} = 5\% \text{ (say)} \qquad [5.2.17]$$

From [5.2.16] and [5.2.17] we get that

$$z \geq .05k + (n+\mu)k$$

$$z \geq (n+\mu+.05)k \qquad [5.2.18]$$

if we have that the rate of depreciation μ is equal to 6% and the growth rate n is equal to 4% we have that [5.2.18] is:

$$z \geq 1.5k$$

if we assume that the capital stock k is given and equal to 10 we have

$$z \geq 15$$

In order to evaluate numerically we consider the following specifications

$$\mu(c,x) = c^{.6}x^{.4}$$

$$f_c(k_c) = k_c^{.8}$$

$$f_v(k_v) = k_v^{.9}$$

now if we replace these expressions in our programming problem [5.2.11] we have

$$\text{Max } c^{.6}x^{.4}$$

subject to

$$c+z \leq k_c^{.8} l_c$$

$$x \leq k_v^{.9} l_v$$

$$k_c l_c + k_v l_v \leq 10$$

$$l_c + l_v \leq 1$$

$$z \geq 15$$

The utility function $u(c,x)$ is continuously twice differentiable and has positive marginal utilities $\frac{\partial u}{\partial c}$ and $\frac{\partial u}{\partial c}$ for all positive c and x; $u(c,x)$ is concave in the sense that the Hessian matrix

$$\underline{H} = \begin{bmatrix} \frac{\partial^2 u}{\partial c^2} & \frac{\partial^2 u}{\partial c \partial x} \\ \frac{\partial^2 u}{\partial c \partial x} & \frac{\partial^2 u}{\partial x^2} \end{bmatrix}$$

is negative semi-definite for all values of c and x.

Private goods are not inferior; the income-consumption curve has a positive slope; these conditions are summarized as:

$$\frac{\partial u}{\partial c} > 0, \ \frac{\partial u}{\partial x} > 0$$

$$\frac{\partial^2 u}{\partial c^2} < 0, \ \frac{\partial^2 u}{\partial x^2} < 0$$

$$|\underline{H}| = \frac{\partial^2 u}{\partial c^2} \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial c \partial x}\right)^2 \geq 0$$

$$\frac{\frac{\partial^2 u}{\partial x^2}}{\frac{\partial u}{\partial x}} - \frac{\frac{\partial^2 u}{\partial c \partial x}}{\frac{\partial u}{\partial x}} < 0, \quad \frac{\frac{\partial^2 u}{\partial c^2}}{\frac{\partial u}{\partial c}} - \frac{\frac{\partial^2 u}{\partial x \partial c}}{\frac{\partial u}{\partial x}} < 0$$

To cast our programming problem into a typical geometric programming formulation let us make the following change of variables.

$c = x_1$

$x = x_2$

$z = x_3$

$k_c = x_4$

$k_v = x_5$

$l_v = x_6$

$l_c = x_7$

In order to maximize the objective function we minimize the inverse of the original objective function. I report I failed to minimize the objective function changing only the sign of it, which produced overflows and underflows in the computer output.

Now our programming problem will be

$$\text{Min } x_1^{-.6} x_2^{-.4}$$

subject to

$$x_1+x_3 \leq x_4^{.8}x_7$$

$$x_2 \leq x_5^{.9}x_7$$

$$x_4x_7+x_5x_6 \leq 10$$

$$x_6+x_7 \leq 1$$

$$x_3 \geq 1.5$$

we had added two restrictions in order to avoid unbounded objective function values, e.g.,

$$x_1^{.6}x_2^{.4} \leq 10$$

$$x_1^{.6}x_2^{.4} \geq .1$$

Finally our programming problem in the typical geometric programming formulation is:

$$\text{Min } x_1^{-.6}x_2^{-.4}$$

subject to

$$x_1x_4^{-.8}x_7^{-1.}+x_3x_4^{-.8}x_7^{-1.} \leq 1$$

$$x_2x_5^{-.9}x_6^{-1} \leq 1$$

$$0.1x_4x_7+0.1x_5x_6 \leq 1$$

$$x_6+x_7 \leq 1$$

$$1.5x_3^{-1} \le 1$$

$$0.1x_1^{.6}x_2^{.4} \le 1$$

$$0.1x_1^{-.6}x_2^{-.4} \le 1$$

We have a geometric programming problem with three degrees of difficulty that is we have eleven terms minus seven variables minus one. We observe that all the expressions are posynomials.

The dual program associated with this problem consists of maximizing the dual function

$$v(\underline{\delta}) = \left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}\left(\frac{1}{\delta_4}\right)^{\delta_4}\left(\frac{0.1}{\delta_5}\right)^{\delta_5}\left(\frac{0.1}{\delta_6}\right)^{\delta_6}\left(\frac{1}{\delta_7}\right)^{\delta_7} \times$$

$$\left(\frac{1}{\delta_8}\right)^{\delta_8}\left(\frac{1}{\delta_9}\right)^{\delta_9}\left(\frac{0.1}{\delta_{10}}\right)^{\delta_{10}}\left(\frac{0.1}{\delta_{11}}\right)^{\delta_{11}}(\delta_2+\delta_3)^{\delta_2+\delta_3}(\delta_4)^{\delta 4} \times$$

$$(\delta_5+\delta_6)^{\delta_5+\delta_6}(\delta_7+\delta_8)^{\delta_7+\delta_8}(\delta_8)^{\delta_8}(\delta_9)^{\delta_9}(\delta_{10})^{\delta_{10}}(\delta_{11})^{\delta_{11}}$$

Simplifying we have

$$\text{Max } v(\underline{\delta}) = \left(\frac{1}{\delta_1}\right)^{\delta_1}\left(\frac{1}{\delta_2}\right)^{\delta_2}\left(\frac{1}{\delta_3}\right)^{\delta_3}\left(\frac{0.1}{\delta_5}\right)^{\delta_5}\left(\frac{0.1}{\delta_6}\right)^{\delta_6}\left(\frac{1}{\delta_7}\right)^{\delta_7}(0.1)^{\delta_{10}} \times$$

$$(0.1)^{\delta}(\delta_2+\delta_3)^{\delta_2+\delta_3}(\delta_5+\delta_6)^{\delta_5+\delta_6}(\delta_7+\delta_8)^{\delta_7+\delta_8}.$$

subject to nonnegative dual variables

$$\delta_1 \ge 0, \delta_2 \ge 0, \delta_3 \ge 0, \delta_4 \ge 0, \delta_5 \ge 0, \delta_6 \ge 0, \delta_7 \ge 0, \delta_8 \ge 0, \delta_9 \ge 0,$$

$$\delta_{10} \ge 0, \delta_{11} \ge 0$$

and the normality and orthogonality conditions give the linear system

$$
\begin{aligned}
\delta_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad &= 1\\
-.6\delta_1+\delta_2 \qquad\qquad\qquad\qquad\qquad\qquad +.6\delta_{10}-.6\delta_{11} &= 0\\
-.4\delta_1 \qquad\quad +\delta_4 \qquad\qquad\qquad\qquad +.4\delta_{10}-.4\delta_{11} &= 0\\
\delta_3 \qquad\qquad\qquad\qquad -\delta_7 \qquad\qquad\qquad &= 0\\
-.8\delta_2-.8\delta_3 \qquad\quad +\delta_5 \qquad\qquad\qquad\qquad\qquad &= 0\\
-.9\delta_4 \qquad\quad +\delta_6 \qquad\qquad\qquad\qquad &= 0\\
-\delta_4 \qquad\quad +\delta_6 \qquad +\delta_8 \qquad\qquad\qquad &= 0\\
-\delta_2 \quad -\delta_3 \qquad\qquad +\delta_5 \qquad\qquad\qquad +\delta_9 \qquad\qquad &= 0
\end{aligned}
$$

The solution of this problem is shown in Appendix C. The primal results are given in Table 5.2.1 and the dual results in Table 5.2.2.

Table 5.2.1. Primal results

Variable	value	
x_1	2.898	c
x_2	2.472	x
x_3	1.5	z
x_4	8.163	k_c
x_5	18.366	k_v
x_6	0.18008501	l_v
x_7	0.81991499	l_c
$g_0(\underline{x})$	0.3677	$g_0(x)$
$u(c,x)$	2.712	$u(c,x)$

Table 5.2.2 Dual results

Variable	Value
δ_1	1.0
δ_2	0.6
δ_3	0.3106
δ_4	0.4
δ_5	0.7285
δ_6	0.36
δ_7	0.3105
δ_8	0.4
δ_9	0.18211
δ_{10}	2.3195×10^{-9}
δ_{11}	3.3127×10^{-8}
$v(\underline{\delta})$	0.3677

We have solved a programming problem where the exponents of the objective function add to one (i.e. $\alpha+\beta=1$) if we increase this sum such that $\alpha+\beta > 1$ we might have a utility function

$$u(c,x) = c^{.8}x^{.4}$$

we also have the upper and lower bound for our objective function as

$$c^{.8}x^{.4} \leq 10$$

$$c^{.8}x^{.4} \geq .1$$

in Table 5.2.3 we report the results and the dual variables in Table 5.2.5.

Table 5.2.3. Primal result, case II

Variable	Value	
x_1	3.2154	c
x_2	2.06484	x
x_3	1.5	z
x_4	8.4998	k_c
x_5	19.034	k_v
x_6	0.1456	l_v
x_7	0.8543	l_c
$g_0(\underline{x})$	0.2939	$g_0(\underline{x})$
u(c,x)	3.41	u(c,x)

Another interesting case (i.e., case III) we have considered, occurs when the change of capital stock $\dot{k}/k$=14% and therefore the investment per capita constraint [5.2.18] will be

$$z \geq (.14 + .04 + .06)\times 10$$

$$z \geq 2.4$$

The results are given in Table 5.2.4, note that we have a decrease in consumption in order to accomplish the investment restrictions, the dual results are given in Table 5.2.5.

Table 5.2.4. Primal results, case III

Variable	Value	
x_1	2.3535	c
x_2	2.0160	x
x_3	2.4	z
x_4	8.49	k_c
x_5	19.1152	k_v
x_6	0.141659	l_v
x_7	0.858341	l_c
$g_0(\underline{x})$	0.4520	$g_0(x)$
$u(c,x)$	2.24	$u(c,x)$

We summarize one result by saying that consumption of private goods (c) in the three cases is higher than that of public goods as is the fraction of labor allocated to produce private goods (l_c).

Table 5.2.5. Dual results

Variable	case II $c^{.8}x^{.4}$	case III $c^{.6}x^{.4}$ and $\dot{k}/k=14\%$
δ_1	1.	1.
δ_2	0.8	0.6
δ_3	0.3732	0.6118
δ_4	0.4	0.4
δ_5	0.9386	.9695
δ_6	0.36	.36
δ_7	0.3732	.6118
δ_8	0.04	0.04
δ_9	0.2346	0.2424
δ_{10}	9.31×10^{-10}	1.29×10^{-9}
δ_{11}	3.17×10^{-8}	1.15×10^{-8}
$\sigma(\underline{\delta})$	0.2939	0.4520

6. SUMMARY AND GENERALIZATIONS

Methods of risk programming essentially convert a probabilistic linear programming model into a nonlinear deterministic form, since most operational decision rules have to be specified in non-random terms. Nevertheless we have already started with a quadratic programming model in section 4.1 and due to the simplifying assumption of an exponential distribution about sales we stayed with a quadratic problem. We have solved a large quadratic programming problem instead of using some simplifying technique as Holt, Modigliani, Muth and Simon and others originally used (67, 68, 66, 34). Our limitation in the production planning model was to deal with coded data, practical research will be applied to Peruvian firms and the results will be reported elsewhere.

In the multi-period investment under uncertainty, we have solved in the nonlinear fashion because we were more lucky than Näslund, that is we have the SUMT algorithm which also allowed us to solve the nonlinear reliability models. We might have a sensitivity analysis of the nonlinear programming model but that computer program was not available at the time we were beginning to program it. The sample distribution approach gives us some insight about how our optimal solution changes due to an increase in sample size.

We have intended to show how to solve nonlinear programming problems using geometric and generalized polynomial programming in order to indicate that we are able to deal with more complicated models of risk programming that will be produced by assuming different distribution functions.

The maximum likelihood principle (2) gives us an open window for further research in risk programming, for example consider the CCP model

$$\text{Max } c'x$$

$$\text{Prob } [Ax \leq b] \geq u$$

$$x \geq 0; \quad 0 < u \leq 1$$

If b_i is assumed a random variable with a normally independent distribution

$$\text{Prob}\left(\frac{b_i - \bar{b}_i}{\sigma_{b_i}} \geq \frac{a_i' x - \bar{b}_i}{\sigma_{b_i}}\right) \geq u_i$$

$$1 - F\left(\frac{a_i' x - \bar{b}_i}{\sigma_i}\right) \geq u_i$$

The cumulative density function $F\left(\frac{a_x' - \bar{b}_i}{\sigma_{b_i}}\right)$ can be expanded by the following approximation (65):

$$F\left(\frac{a_i' x - \bar{b}_i}{\sigma_{b_i}}\right) \simeq \frac{1}{2} + \frac{1}{\sqrt{2\Pi}} \left(\frac{a_i' x - \bar{b}_i}{\sigma_{b_i}}\right) - \frac{1}{6\sqrt{2\Pi}} \left(\frac{a_i' x - \bar{b}_i}{\sigma_{b_i}}\right)^3$$

Let us define $\xi_i = \frac{a_i'x-\bar{b}_i}{\sigma_{b_i}}$

we have a transformed deterministic problem such as

$$\text{Max } c'x + \Sigma \ln [1-F(\xi_i)]$$

subject to

$$x \geq 0$$

or

$$\text{Max } c'x + \sum_{i=1}^{m} \ln\left(\frac{1}{2} - \frac{1}{\sqrt{2\Pi}} \xi_i + \frac{1}{6\sqrt{2\Pi}} \xi_i^3\right)$$

subject to

$$x \geq 0$$

So far we have found it difficult to solve this class of nonlinear programming problems, but the day is not far when we can solve these problems. Further research will be continued by me in this direction and I would attempt to apply these optimization techniques to various industrial and national planning problems in Peru.

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9. APPENDIX A: PRODUCTION PLANNING - A COMPUTER PROGRAM

```
****************************************************************************************************************

FORTRAN IV G LEVEL 1, MOD 4              MAIN               DATE = 69282          15/00/27              PAGE 0001

 0001                DIMENSION A(100,100),PB(100)
             C       THE BIG MATRIX GOES TO ZERO
 0002                DATA A/10000*0./, PB/100*0./
             C       READING PARAMETERS
 0003                XLAM = ASALE - SALMIN
             C
             C                            THE QUADRATIC  MATRIX
             C
 0004                L2 = 2*NTIME
 0005                L3 = L2 + NTIME
 0006                L4 = L3 + NTIME
 0007                L5 = L4 + NTIME
 0008                L6 = L5 + NTIME
 0009                L7 = L6 + NTIME
 0010                L8 = L7 + NTIME
             C       GENERATING THE SUBMATRIX OF WORK FORCE
 0011                NFINAL = NTIME - 1
 0012                DO 3 K = 1,NFINAL
 0013                A(K,K) = (4.*C2 + 2.*C3*C4**2)*(-W1)
 0014                L = K + 1
 0015                A(K,L) = (-2.*C2)*(-W1)
 0016              3 A(L,K) = A(K,L)
 0017                A(NTIME,NTIME) = (2.*C2 + 2.*C3*C4**2)*(-W1)
             C       GENERATING THE SUBMATRICES OF WORK-PRODUCTION AND PRODUCTION ITSELF
 0018                J= NTIME
 0019                DO 4  I = 1, NTIME
 0020                J = J + 1
 0021                A(I,J) =  (-2.*C3*C4)*(-W1)
 0022                A(J,I) = A(I,J)
 0023              4 A(J,J) =  (2.*C3)*(-W1)
             C       GENERATING THE SUBMATRIX OF INVENTORY
 0024                L = L2 + 1
 0025                DO 5  I = L,L3
 0026              5 A(I,I) = (2.*C7)*(-W1)
             C       GENERATING THE SUBMATRIX OF RELIABILITY INDEX
 0027                L = L3 + 1
 0028                DO 6  I = L,L4
 0029              6 A(I,I) = 0.
             C       GENERATING THE PB VECTOR
 0030                PB(1) = (C1 - C6 - 2.*C2*WORI)*W1
 0031                DO 7  I= 2,NTIME
 0032              7 PB(I) =(C1-C2)*W1
 0033                II = NTIME + 1
 0034                DO 8  K = II,L2
 0035              8 PB(K) = C5*W1
 0036                II = L2 + 1
 0037                DO 9  K = II,L3
 0038              9 PB(K) = -2.*C7*(C8 + C9*ASALE)*W1
 0039                II = L3 + 1
 0040                DO 10 K = II,L4
 0041             10 PB(K) = -W2  .
             C       GENERATING THE RIGHT HAND SIDE VECTOR
             C            WORK FORCE RESTRICTION
```

```
            II = L4 + 1
            DO 11 K = II,L5
      11 PB(K) = WORK
            K = L5 + 1
            PB(K) =  SALMIN - XINVI
            IF(PB(K)) 12,13,13
      12 PB(K) = -PB(K)
            I333  = 1
            GO TO 14
      13 I333 = 0
      14 K = L5 + 2
            DO 15 I =K,L6
      15 PB(I) = SALMIN
            II = L6 + 1
            DO 16 K = II,L7
      16 PB(K) = YLOW
            II = L7 + 1
            DO 17  K = II,L8
      17 PB(K) = UPPY
C                                  GENERATING THE RESTRICTION SUBMATRICES
C           THE WORK FORCE RESTRICTION SUBMATRIX
            I = L4
            DO 18  J = 1,NTIME
            I = I + 1
      18 A(I,J) =  1.
C           THE PRODUCTION CONSTRAINT MATRIX
            II = L5 + 1
            J  = NTIME
            DO 19  I = II,L6
            J = J + 1
      19 A(I,J) = 1.
            J = L2
            I8 = L5 + 2
            DO 20 I = I8,L6
            J = J + 1
      20 A(I,J) = 1.
            J = L3
            DO 21 I = II,L6
            J = J + 1
      21 A(I,J) = -XLAM
            IK = L3 + 1
C                 LOWER  AND  UPPER  BOUND  OF  Y
            J = L2
            II = L6 + 1
            DO 22  I= II,L7
            J = J + 1
      22 A(I,J) = 1.
            II = L7 + 1
            J = L3
            IF(YLOW.NE.0) GO TO 230
            QQ = 0.
            GO TO 232
     230 QQ=1.
```

```
    232 DO 23  I= II,L8
        J = J + 1
     23 A(I,J) = QQ
C       THE ARTIFICIAL MATRIX
        IN = L4 + 1
        DO 24  I = IN,L7
     24 A(I,I) = -1.
        I = L5 + 1
        IF(I333 - 1) 27, 25, 27
     25 A(I,I) = 0.
        ISTAR = L5 + 1
        DO 26  I=NTIME,L4
     26 A(ISTAR,I) = -A(ISTAR,I)
        GO TO 28
     27 A(I,I) = -1
     28 CONTINUE
        IF(YLOW.NE.0) GO TO 281
        IQ = L6 + 1
        DO 280 I = IQ,L7
    280 A(I,I) = C.
    281 CONTINUE
C
C                  ************************************
        WRITE(19,1919)  ((A(I,J),J=1,L4),I=1,L4),  (PB(I) ,I=1,L4)
   1919 FORMAT(4X,5D15.8)
C                             SCALING
C                  ********************
        DO 5050 I=1,L4
        PB(I)= PB(I)/10.
        DO 5050 J = 1,L4
   5050 A(I,J) =A(I,J)/100.
        KX=L4 +1
        DO 7070 J =1,L4
        DO 7070 I =KX,L8
   7070 A(I,J) =A(I,J)/10.
C             ----------------------------------------------
C                           THE OUTPUT GENERATION
C
        IEND = L4 + 100
        J= 101
        WRITE(IND,29) (K, K=J,IEND)
     29 FORMAT(          'INPUT.      HMMS  '/
       1SCAN.'/'ROW.ID       HMMS  '/,(12X,'R',I3))
C       COMPLETING THE QUADRATIC MATRIX
        JF = L3 + 99
        WRITE(IND,31)  (K, K=J,JF)
     31 FORMAT(12X,'A',I3)
        IF(I333.NE.0) GO TO 32
        JF = JF +1
        WRITE(IND,31) JF
     32 IEND = L4 + 101
        IFIN = L5 + 100
        WRITE(IND,33) (K, K=IEND,IFIN)
```

```
      33 FORMAT(11X,'-R',I3)
         K = IFIN + 1
         IF(I333 - 1) 34, 35, 34
      34 WRITE(IND,33)  K
         GO TO 37
      35 WRITE(IND,36)  K
      36 FORMAT(11X,'+R',I3)
      37 IEND = IFIN + 2
         IF(YLOW.EQ.0) GO TO 38
         GO TO(39, 38),  ICOND
      38 IFIN = L6 + 100
         GO TO 40
      39 IFIN = L7 + 100
      40 WRITE(IND,33)  (K,K=IEND,IFIN)
         IEND = IFIN + 1
         IFIN = L8 + 100
         WRITE(IND,36)  (K, K=IEND,IFIN)
      42 WRITE(IND,43)
      43 FORMAT('MATRIX')
         DO 46 J = 1,L8
         KCOL = 100 + J
         DO 46 I = 1,L6
         KROW = 100 + I
         IF(A(I,J))  44, 46, 44
      44 WRITE(IND,45)  KCOL,KROW, A(I,J)
      45 FORMAT(6X,'C',I3,2X,'R',I3,2X,F12.6)
      46 CONTINUE
         WRITE(IND,47)
      47 FORMAT('FIRSTB')
         DO 50 J = 1,L8
         KCOL = 100 + J
         IF(PB(J)) 48,50,48
      48 WRITE(IND,49)  KCOL, PB(J)
      49 FORMAT(6X,'PB.',3X,'R',I3,2X,F12.6)
      50 CONTINUE
         WRITE(IND,51)
      51 FORMAT('ENDATA'/'MODEL.'/'MAX...'/'OUTPUT'/'CHECK'/'INVERT'/'OUTPU
        1T'/'ENDJOB')
         STOP
         END
```

```
*****************************************************************************************************************************

                                   ISU 360 Q.P.-DISK VERSION

INVERT FREQ. PIVOT TOL.  PRICE TOL.  FEASB.TOL.  ARTFL.TOL.  ELEMT.TOL.,
    9999     .000100000  .000100000  .001000000  .000100000  .000001000
```

INPUT. HMMS

OCT 06, 1969

SCAN.

	ROW.ID	HMMS
0	R101	0.0
0	R102	0.0
0	R103	0.0
0	R104	0.0
0	R105	0.0
0	R106	0.0
0	R107	0.0
0	R108	0.0
0	R109	0.0
0	R110	0.0
0	R111	0.0
0	R112	0.0
0	R113	0.0
0	R114	0.0
0	R115	0.0
0	R116	0.0
0	R117	0.0
0	R118	0.0
0	R119	0.0
0	R120	0.0
0	R121	0.0
0	R122	0.0
0	R123	0.0
0	R124	0.0
0	R125	0.0
0	R126	0.0
0	R127	0.0
0	R128	0.0
0	R129	0.0
0	R130	0.0
0	R131	0.0
0	R132	0.0
0	R133	0.0
0	R134	0.0
0	R135	0.0
0	R136	0.0
0	R137	0.0
0	R138	0.0
0	R139	0.0
0	R140	0.0
0	R141	0.0
0	R142	0.0
0	R143	0.0
0	R144	0.0
0	R145	0.0
0	R146	0.0
0	R147	0.0
0	R148	0.0
0	A101	0.0
0	A102	0.0
0	A103	0.0
0	A104	0.0
0	A105	0.0
0	A106	0.0
0	A107	0.0
0	A108	0.0

C	A109	0.0
C	A110	0.0
C	A111	0.0
C	A112	0.0
C	A113	0.0
C	A114	0.0
C	A115	0.0
C	A116	0.0
C	A117	0.0
C	A118	0.0
C	A119	0.0
C	A120	0.0
C	A121	0.0
C	A122	0.0
C	A123	0.0
C	A124	0.0
C	A125	0.0
C	A126	0.0
C	A127	0.0
C	A128	0.0
C	A129	0.0
C	A130	0.0
C	A131	0.0
C	A132	0.0
C	A133	0.0
C	A134	0.0
C	A135	0.0
C	-R149	0.0
C	-R150	0.0
C	-R151	0.0
C	-R152	0.0
C	-R153	0.0
C	-R154	0.0
C	-R155	0.0
C	-R156	0.0
C	-R157	0.0
C	-R158	0.0
C	-R159	0.0
C	-R160	0.0
C	+R161	0.0
C	-R162	0.0
C	-R163	0.0
C	-R164	0.0
C	-R165	0.0
C	-R166	0.0
C	-R167	0.0
C	-R168	0.0
C	-R169	0.0
C	-R170	0.0
C	-R171	0.0
C	-R172	0.0
C	-R173	0.0
C	-R174	0.0
C	-R175	0.0
C	-R176	0.0
C	-R177	0.0
C	-R178	0.0
C	-R179	0.0
C	-R180	0.0
C	-R181	0.0
C	-R182	0.0

```
0              -R183     0.0
0              -R184     0.0
0              +R185     0.0
0              +R186     0.0
0              +R187     0.0
0              +R188     0.0
0              +R189     0.0
0              +R190     0.0
0              +R191     0.0
0              +R192     0.0
0              +R193     0.0
0              +R194     0.0
0              +R195     0.0
0              +R196     0.0

        MATRIX
0              C101  R101     -2.160474
0              C101  R102      1.028798
0              C101  R113      0.018144
0              C101  R149      0.100000
0              C102  R101      1.028798
0              C102  R102     -2.160474
0              C102  R103      1.028798
0              C102  R114      0.018144
0              C102  R150      0.100000
0              C103  R102      1.028798
0              C103  R103     -2.160474
0              C103  R104      1.028798
0              C103  R115      0.018144
0              C103  R151      0.100000
0              C104  R103      1.028798
0              C104  R104     -2.160474
0              C104  R105      1.028798
0              C104  R116      0.018144
0              C104  R152      0.100000
0              C105  R104      1.028798
0              C105  R105     -2.160474
0              C105  R106      1.028798
0              C105  R117      0.018144
0              C105  R153      0.100000
0              C106  R105      1.028798
0              C106  R106     -2.160474
0              C106  R107      1.028798
0              C106  R118      0.018144
0              C106  R154      0.100000
0              C107  R106      1.028798
0              C107  R107     -2.160474
0              C107  R108      1.028798
0              C107  R119      0.018144
0              C107  R155      0.100000
0              C108  R107      1.028798
0              C108  R108     -2.160474
0              C108  R109      1.028798
0              C108  R120      0.018144
0              C108  R156      0.100000
0              C109  R108      1.028798
0              C109  R109     -2.160474
0              C109  R110      1.028798
0              C109  R121      0.018144
0              C109  R157      0.100000
0              C110  R109      1.028798
```

0	C110	R110	-2.160474
0	C110	R111	1.028798
0	C110	R122	0.018144
0	C110	R158	0.100000
0	C111	R110	1.028798
0	C111	R111	-2.160474
0	C111	R112	1.028798
0	C111	R123	0.018144
0	C111	R159	0.100000
0	C112	R111	1.028798
0	C112	R112	-1.131676
0	C112	R124	0.018144
0	C112	R160	0.100000
0	C113	R101	0.018144
0	C113	R113	-0.003200
0	C113	R161	-0.100000
0	C114	R102	0.018144
0	C114	R114	-0.003200
0	C114	R162	0.100000
0	C115	R103	0.018144
0	C115	R115	-0.003200
0	C115	R163	0.100000
0	C116	R104	0.018144
0	C116	R116	-0.003200
0	C116	R164	0.100000
0	C117	R105	0.018144
0	C117	R117	-0.003200
0	C117	R165	0.100000
0	C118	R106	0.018144
0	C118	R118	-0.003200
0	C118	R166	0.100000
0	C119	R107	0.018144
0	C119	R119	-0.003200
0	C119	R167	0.100000
0	C120	R108	0.018144
0	C120	R120	-0.003200
0	C120	R168	0.100000
0	C121	R109	0.018144
0	C121	R121	-0.003200
0	C121	R169	0.100000
0	C122	R110	0.018144
0	C122	R122	-0.003200
0	C122	R170	0.100000
0	C123	R111	0.018144
0	C123	R123	-0.003200
0	C123	R171	0.100000
0	C124	R112	0.018144
0	C124	R124	-0.003200
0	C124	R172	0.100000
0	C125	R125	-0.001320
0	C125	R162	0.100000
0	C126	R126	-0.001320
0	C126	R163	0.100000
0	C127	R127	-0.001320
0	C127	R164	0.100000
0	C128	R128	-0.001320
0	C128	R165	0.100000
0	C129	R129	-0.001320
0	C129	R166	0.100000
0	C130	R130	-0.001320
0	C130	R167	0.100000

0	C131	R131	-0.001320
0	C131	R168	0.100000
0	C132	R132	-0.001320
0	C132	R169	0.100000
0	C133	R133	-0.001320
0	C133	R170	0.100000
0	C134	R134	-0.001320
0	C134	R171	0.100000
0	C135	R135	-0.001320
0	C135	R172	0.100000
0	C136	R136	-0.001320
0	C137	R161	20.000000
0	C137	R173	0.100000
0	C137	R185	0.100000
0	C138	R162	-20.000000
0	C138	R174	0.100000
0	C138	R186	0.100000
0	C139	R163	-20.000000
0	C139	R175	0.100000
0	C139	R187	0.100000
0	C140	R164	-20.000000
0	C140	R176	0.100000
0	C140	R188	0.100000
0	C141	R165	-20.000000
0	C141	R177	0.100000
0	C141	R189	0.100000
0	C142	R166	-20.000000
0	C142	R178	0.100000
0	C142	R190	0.100000
0	C143	R167	-20.000000
0	C143	R179	0.100000
0	C143	R191	0.100000
0	C144	R168	-20.000000
0	C144	R180	0.100000
0	C144	R192	0.100000
0	C145	R169	-20.000000
0	C145	R181	0.100000
0	C145	R193	0.100000
0	C146	R170	-20.000000
0	C146	R182	0.100000
0	C146	R194	0.100000
0	C147	R171	-20.000000
0	C147	R183	0.100000
0	C147	R195	0.100000
0	C148	R172	-20.000000
0	C148	R184	0.100000
0	C149	R196	0.100000
0	C149	R149	-1.000000
0	C150	R150	-1.000000
0	C151	R151	-1.000000
0	C152	R152	-1.000000
0	C153	R153	-1.000000
0	C154	R154	-1.000000
0	C155	R155	-1.000000
0	C156	R156	-1.000000
0	C157	R157	-1.000000
0	C158	R158	-1.000000
0	C159	R159	-1.000000
0	C160	R160	-1.000000
0	C162	R162	-1.000000
0	C163	R163	-1.000000

0	C164	R164	-1.000000
0	C165	R165	-1.000000
0	C166	R166	-1.000000
0	C167	R167	-1.000000
0	C168	R168	-1.000000
0	C169	R169	-1.000000
0	C170	R170	-1.000000
0	C171	R171	-1.000000
0	C172	R172	-1.000000
0	C173	R173	-1.000000
0	C174	R174	-1.000000
0	C175	R175	-1.000000
0	C176	R176	-1.000000
0	C177	R177	-1.000000
0	C178	R178	-1.000000
0	C179	R179	-1.000000
0	C180	R180	-1.000000
0	C181	R181	-1.000000
0	C182	R182	-1.000000
0	C183	R183	-1.000000
0	C184	R184	-1.000000

FIRSTB

0	PB.	R101	-921.199463
0	PB.	R102	22.055969
0	PB.	R103	22.055969
0	PB.	R104	22.055969
0	PB.	R105	22.055969
0	PB.	R106	22.055969
0	PB.	R107	22.055969
0	PB.	R108	22.055969
0	PB.	R109	22.055969
0	PB.	R110	22.055969
0	PB.	R111	22.055969
0	PB.	R112	22.055969
0	PB.	R113	4.095999
0	PB.	R114	4.095999
0	PB.	R115	4.095999
0	PB.	R116	4.095999
0	PB.	R117	4.095999
0	PB.	R118	4.095999
0	PB.	R119	4.095999
0	PB.	R120	4.095999
0	PB.	R121	4.095999
0	PB.	R122	4.095999
0	PB.	R123	4.095999
0	PB.	R124	4.095999
0	PB.	R125	-4.223996
0	PB.	R126	-4.223996
0	PB.	R127	-4.223996
0	PB.	R128	-4.223996
0	PB.	R129	-4.223996
0	PB.	R130	-4.223996
0	PB.	R131	-4.223996
0	PB.	R132	-4.223996
0	PB.	R133	-4.223996
0	PB.	R134	-4.223996
0	PB.	R135	-4.223996
0	PB.	R136	-4.223996
0	PB.	R137	-0.020000
0	PB.	R138	-0.020000

```
0          PB.   R139    -0.020000
0          PB.   R140    -0.020000
0          PB.   R141    -0.020000
0          PB.   R142    -0.020000
0          PB.   R143    -0.020000
0          PB.   R144    -0.020000
0          PB.   R145    -0.020000
0          PB.   R146    -0.020000
0          PB.   R147    -0.020000
0          PB.   R148    -0.020000
0          PB.   R149    75.000000
0          PB.   R150    75.000000
0          PB.   R151    75.000000
0          PB.   R152    75.000000
0          PB.   R153    75.000000
0          PB.   R154    75.000000
0          PB.   R155    75.000000
0          PB.   R156    75.000000
0          PB.   R157    75.000000
0          PB.   R158    75.000000
0          PB.   R159    75.000000
0          PB.   R160    75.000000
0          PB.   R161    20.000000
0          PB.   R162   300.000000
0          PB.   R163   300.000000
0          PB.   R164   300.000000
0          PB.   R165   300.000000
0          PB.   R166   300.000000
0          PB.   R167   300.000000
0          PB.   R168   300.000000
0          PB.   R169   300.000000
0          PB.   R170   300.000000
0          PB.   R171   300.000000
0          PB.   R172   300.000000
0          PB.   R173     2.659260
0          PB.   R174     2.659260
0          PB.   R175     2.659260
0          PB.   R176     2.659260
0          PB.   R177     2.659260
0          PB.   R178     2.659260
0          PB.   R179     2.659260
0          PB.   R180     2.659260
0          PB.   R181     2.659260
0          PB.   R182     2.659260
0          PB.   R183     2.659260
0          PB.   R184     2.659260
0          PB.   R185     2.659260
0          PB.   R186     2.659260
0          PB.   R187     2.659260
0          PB.   R188     2.659260
0          PB.   R189     2.659260
0          PB.   R190     2.659260
0          PB.   R191     2.659260
0          PB.   R192     2.659260
0          PB.   R193     2.659260
0          PB.   R194     2.659260
0          PB.   R195     2.659260
0          PB.   R196     2.659260
     ENDATA
```

NUMBER ROWS IN A	VARIABLES	RHS	DENSITY
48	83	1	294

AGENDUM TIME WAS 0.183 MINUTES

```
MODEL.

ROW IDENT-NAME...  COL NAMES...  MATRIX ELEMENTS..(ROW NUMBER; COEFFICIENT)..........

 1 40001 R101...  10001 C101...    1   -2.1605    2    1.0288   13    0.0181   84   0.1000
 2 40002 R102...  10002 C102...    1    1.0288    2   -2.1605    3    1.0288   14   0.0181   85   0.1000
 3 40003 R103...  10003 C103...    2    1.0288    3   -2.1605    4    1.0288   15   0.0181   86   0.1000
 4 40004 R104...  10004 C104...    3    1.0288    4   -2.1605    5    1.0288   16   0.0181   87   0.1000
 5 40005 R105...  10005 C105...    4    1.0288    5   -2.1605    6    1.0288   17   0.0181   88   0.1000
 6 40006 R106...  10006 C106...    5    1.0288    6   -2.1605    7    1.0288   18   0.0181   89   0.1000
 7 40007 R107...  10007 C107...    6    1.0288    7   -2.1605    8    1.0288   19   0.0181   90   0.1000
 8 40008 R108...  10008 C108...    7    1.0288    8   -2.1605    9    1.0288   20   0.0181   91   0.1000
 9 40009 R109...  10009 C109...    8    1.0288    9   -2.1605   10    1.0288   21   0.0181   92   0.1000
10 40010 R110...  10010 C110...    9    1.0288   10   -2.1605   11    1.0288   22   0.0181   93   0.1000
11 40011 R111...  10011 C111...   10    1.0288   11   -2.1605   12    1.0288   23   0.0181   94   0.1000
12 40012 R112...  10012 C112...   11    1.0288   12   -1.1317   24    0.0181   95   0.1000
13 40013 R113...  10013 C113...    1    0.0181   13   -0.0032   96   -0.1000
14 40014 R114...  10014 C114...    2    0.0181   14   -0.0032   97    0.1000
15 40015 R115...  10015 C115...    3    0.0181   15   -0.0032   98    0.1000
16 40016 R116...  10016 C116...    4    0.0181   16   -0.0032   99    0.1000
17 40017 R117...  10017 C117...    5    0.0181   17   -0.0032  100    0.1000
18 40018 R118...  10018 C118...    6    0.0181   18   -0.0032  101    0.1000
19 40019 R119...  10019 C119...    7    0.0181   19   -0.0032  102    0.1000
20 40020 R120...  10020 C120...    8    0.0181   20   -0.0032  103    0.1000
21 40021 R121...  10021 C121...    9    0.0181   21   -0.0032  104    0.1000
22 40022 R122...  10022 C122...   10    0.0181   22   -0.0032  105    0.1000
23 40023 R123...  10023 C123...   11    0.0181   23   -0.0032  106    0.1000
24 40024 R124...  10024 C124...   12    0.0181   24   -0.0032  107    0.1000
25 40025 R125...  10025 C125...   25   -0.0013   97    0.1000
26 40026 R126...  10026 C126...   26   -0.0013   98    0.1000
27 40027 R127...  10027 C127...   27   -0.0013   99    0.1000
28 40028 R128...  10028 C128...   28   -0.0013  100    0.1000
29 40029 R129...  10029 C129...   29   -0.0013  101    0.1000
30 40030 R130...  10030 C130...   30   -0.0013  102    0.1000
31 40031 R131...  10031 C131...   31   -0.0013  103    0.1000
32 40032 R132...  10032 C132...   32   -0.0013  104    0.1000
33 40033 R133...  10033 C133...   33   -0.0013  105    0.1000
34 40034 R134...  10034 C134...   34   -0.0013  106    0.1000
35 40035 R135...  10035 C135...   35   -0.0013  107    0.1000
36 40036 R136...  10036 C136...   36   -0.0013
37 40037 R137...  10037 C137...   96   20.0000  108    0.1000  120    0.1000
38 40038 R138...  10038 C138...   97  -20.0000  109    0.1000  121    0.1000
39 40039 R139...  10039 C139...   98  -20.0000  110    0.1000  122    0.1000
40 40040 R140...  10040 C140...   99  -20.0000  111    0.1000  123    0.1000
41 40041 R141...  10041 C141...  100  -20.0000  112    0.1000  124    0.1000
42 40042 R142...  10042 C142...  101  -20.0000  113    0.1000  125    0.1000
43 40043 R143...  10043 C143...  102  -20.0000  114    0.1000  126    0.1000
44 40044 R144...  10044 C144...  103  -20.0000  115    0.1000  127    0.1000
45 40045 R145...  10045 C145...  104  -20.0000  116    0.1000  128    0.1000
46 40046 R146...  10046 C146...  105  -20.0000  117    0.1000  129    0.1000
47 40047 R147...  10047 C147...  106  -20.0000  118    0.1000  130    0.1000
48 40048 R148...  10048 C148...  107  -20.0000  119    0.1000  131    0.1000
49 40049 A101...  10049 C149...   84   -1.0000
50 40050 A102...  10050 C150...   85   -1.0000
51 40051 A103...  10051 C151...   86   -1.0000
52 40052 A104...  10052 C152...   87   -1.0000
53 40053 A105...  10053 C153...   88   -1.0000
54 40054 A106...  10054 C154...   89   -1.0000
55 40055 A107...  10055 C155...   90   -1.0000
56 40056 A108...  10056 C156...   91   -1.0000
57 40057 A109...  10057 C157...   92   -1.0000
```

58	40058	A110...	10058	C158...	93	-1.0000								
59	40059	A111...	10059	C159...	94	-1.0000								
60	40060	A112...	10060	C160...	95	-1.0000								
61	40061	A113...	10061	C162...	97	-1.0000								
62	40062	A114...	10062	C163...	98	-1.0000								
63	40063	A115...	10063	C164...	99	-1.0000								
64	40064	A116...	10064	C165...	100	-1.0000								
65	40065	A117...	10065	C166...	101	-1.0000								
66	40066	A118...	10066	C167...	102	-1.0000								
67	40067	A119...	10067	C168...	103	-1.0000								
68	40068	A120...	10068	C169...	104	-1.0000								
69	40069	A121...	10069	C170...	105	-1.0000								
70	40070	A122...	10070	C171...	106	-1.0000								
71	40071	A123...	10071	C172...	107	-1.0000								
72	40072	A124...	10072	C173...	108	-1.0000								
73	40073	A125...	10073	C174...	109	-1.0000								
74	40074	A126...	10074	C175...	110	-1.0000								
75	40075	A127...	10075	C176...	111	-1.0000								
76	40076	A128...	10076	C177...	112	-1.0000								
77	40077	A129...	10077	C178...	113	-1.0000								
78	40078	A130...	10078	C179...	114	-1.0000								
79	40079	A131...	10079	C180...	115	-1.0000								
80	40080	A132...	10080	C181...	116	-1.0000								
81	40081	A133...	10081	C182...	117	-1.0000								
82	40082	A134...	10082	C183...	118	-1.0000								
83	40083	A135...	10083	C184...	119	-1.0000								
84	20084	R149...	30084	R149...	49	1.0000	1	-0.1000						
85	20085	R150...	30085	R150...	50	1.0000	2	-0.1000						
86	20086	R151...	30086	R151...	51	1.0000	3	-0.1000						
87	20087	R152...	30087	R152...	52	1.0000	4	-0.1000						
88	20088	R153...	30088	R153...	53	1.0000	5	-0.1000						
89	20089	R154...	30089	R154...	54	1.0000	6	-0.1000						
90	20090	R155...	30090	R155...	55	1.0000	7	-0.1000						
91	20091	R156...	30091	R156...	56	1.0000	8	-0.1000						
92	20092	R157...	30092	R157...	57	1.0000	9	-0.1000						
93	20093	R158...	30093	R158...	58	1.0000	10	-0.1000						
94	20094	R159...	30094	R159...	59	1.0000	11	-0.1000						
95	20095	R160...	30095	R160...	60	1.0000	12	-0.1000						
96	20096	R161...	30096	R161...	37	-20.0000	13	0.1000						
97	20097	R162...	30097	R162...	61	1.0000	38	20.0000	25	-0.1000	14	-0.1000		
98	20098	R163...	30098	R163...	62	1.0000	39	20.0000	26	-0.1000	15	-0.1000		
99	20099	R164...	30099	R164...	63	1.0000	40	20.0000	27	-0.1000	16	-0.1000		
100	20100	R165...	30100	R165...	64	1.0000	41	20.0000	28	-0.1000	17	-0.1000		
101	20101	R166...	30101	R166...	65	1.0000	42	20.0000	29	-0.1000	18	-0.1000		
102	20102	R167...	30102	R167...	66	1.0000	43	20.0000	30	-0.1000	19	-0.1000		
103	20103	R168...	30103	R168...	67	1.0000	44	20.0000	31	-0.1000	20	-0.1000		
104	20104	R169...	30104	R169...	68	1.0000	45	20.0000	32	-0.1000	21	-0.1000		
105	20105	R170...	30105	R170...	69	1.0000	46	20.0000	33	-0.1000	22	-0.1000		
106	20106	R171...	30106	R171...	70	1.0000	47	20.0000	34	-0.1000	23	-0.1000		
107	20107	R172...	30107	R172...	71	1.0000	48	20.0000	35	-0.1000	24	-0.1000		
108	20108	R173...	30108	R173...	72	1.0000	37	-0.1000						
109	20109	R174...	30109	R174...	73	1.0000	38	-0.1000						
110	20110	R175...	30110	R175...	74	1.0000	39	-0.1000						
111	20111	R176...	30111	R176...	75	1.0000	40	-0.1000						
112	20112	R177...	30112	R177...	76	1.0000	41	-0.1000						
113	20113	R178...	30113	R178...	77	1.0000	42	-0.1000						
114	20114	R179...	30114	R179...	78	1.0000	43	-0.1000						
115	20115	R180...	30115	R180...	79	1.0000	44	-0.1000						
116	20116	R181...	30116	R181...	80	1.0000	45	-0.1000						
117	20117	R182...	30117	R182...	81	1.0000	46	-0.1000						
119	20119	R183...	30119	R183...	82	1.0000	47	-0.1000						

119	20119	R184...	30119	R184...	93	1.0000	48	-0.1000
120	20120	R185...	30120	R185...	37	-0.1000		
121	20121	R186...	30121	R186...	38	-0.1000		
122	20122	R187...	30122	R187...	39	-0.1000		
123	20123	R188...	30123	R188...	40	-0.1000		
124	20124	R189...	30124	R189...	41	-0.1000		
125	20125	R190...	30125	R190...	42	-0.1000		
126	20126	R191...	30126	R191...	43	-0.1000		
127	20127	R192...	30127	R192...	44	-0.1000		
128	20128	R193...	30128	R193...	45	-0.1000		
129	20129	R194...	30129	R194...	46	-0.1000		
130	20130	R195...	30130	R195...	47	-0.1000		
131	20131	R196...	30131	R196...	48	-0.1000		

AGENDUM TIME WAS 0.022 MINUTES

MAX... PRICING FRACTION 1.000

ITERATION ACCOUNTING

ITR	ENTER	BASIS	LEAVE	ZJ-CJ S	B	NB	INFEAS			
1	10001	84	20084	-35	0	0	4156.91112	43	101	-0.10000
2	30084	1	40001	-35	1	84	4156.91112	46	101	-0.10000
3	10002	85	20085	0	0	0	4081.91112	54	101	-0.10000
4	30085	2	40002	0	2	85	4081.91112	57	101	-0.10000
5	10003	86	20086	0	0	0	4006.91112	65	101	-0.10000
6	30086	3	40003	0	3	86	4006.91112	68	101	-0.10000
7	10004	87	20087	0	0	0	3931.91112	76	101	-0.10000
8	30087	4	40004	0	4	87	3931.91112	79	101	-0.10000
9	10005	88	20088	0	0	0	3856.91112	87	101	-0.10000
10	30088	5	40005	0	5	88	3856.91112	90	101	-0.10000
11	10006	89	20089	0	0	0	3781.91112	98	101	-0.10000
12	30089	6	40006	0	6	89	3781.91112	101	101	-0.10000
13	10007	90	20090	0	0	0	3706.91112	109	101	-0.10000
14	30090	7	40007	0	7	90	3706.91112	112	101	-0.10000
15	10008	91	20091	-28	0	0	3631.91112	120	101	-0.10000
16	30091	8	40008	-28	8	91	3631.91112	123	101	-0.10000
17	10009	92	20092	0	0	0	3556.91112	131	101	-0.10000
18	30092	9	40009	0	9	92	3556.91112	134	101	-0.10000
19	10010	93	20093	0	0	0	3481.91112	142	101	-0.10000
20	30093	10	40010	0	10	93	3481.91112	145	101	-0.10000
21	10011	94	20094	0	0	0	3406.91112	153	101	-0.10000
22	30094	11	40011	0	11	94	3406.91112	156	101	-0.10000
23	10012	95	20095	0	0	0	3331.91112	163	101	-0.10000
24	30095	12	40012	0	12	95	3331.91112	166	101	-0.10000
25	10014	97	20097	0	0	0	3031.91112	172	101	-0.10000
26	30097	14	40014	0	14	97	3031.91112	177	101	-0.10000
27	10015	98	20098	0	0	0	2731.91112	183	101	-0.10000
28	30098	15	40015	0	15	98	2731.91112	188	101	-0.10000
29	10016	99	20099	-21	0	0	2431.91112	194	101	-0.10000
30	30099	16	40016	-21	16	99	2431.91112	199	101	-0.10000
31	10017	100	20100	0	0	0	2131.91112	205	101	-0.10000
32	30100	17	40017	0	17	100	2131.91112	210	101	-0.10000
33	10018	101	20101	0	0	0	1831.91112	216	101	-0.10000
34	30101	18	40018	0	18	101	1831.91112	221	101	-0.10000
35	10019	102	20102	0	0	0	1531.91112	227	101	-0.10000
36	30102	19	40019	0	19	102	1531.91112	232	101	-0.10000
37	10020	103	20103	0	0	0	1231.91112	238	101	-0.10000
38	30103	20	40020	0	20	103	1231.91112	243	101	-0.10000
39	10021	104	20104	0	0	0	931.91112	249	101	-0.10000
40	30104	21	40021	0	21	104	931.91112	254	101	-0.10000
41	10022	105	20105	0	0	0	631.91112	260	101	-0.10000
42	30105	22	40022	0	22	105	631.91112	265	101	-0.10000
43	10023	106	20106	-14	0	0	331.91112	271	101	-0.10000
44	30106	23	40023	-14	23	106	331.91112	276	101	-0.10000
45	10024	107	20107	0	0	0	31.91112	282	101	-0.10000
46	30107	24	40024	0	24	107	31.91112	287	101	-0.10000
47	10037	96	20096	0	0	0	31.81112	292	101	-0.10000
48	30096	37	40037	0	37	96	31.81112	295	101	-0.10000
49	10038	109	20109	-12	0	0	29.15186	306	101	-0.10000
50	30109	38	40038	-12	38	109	29.15186	309	101	-0.10000
51	10039	110	20110	1	0	0	26.49260	320	101	-0.10000
52	30110	39	40039	1	39	110	26.49260	323	101	-0.10000
53	10040	111	20111	1	0	0	23.83334	334	101	-0.10000
54	30111	40	40040	1	40	111	23.83334	337	101	-0.10000

55	10041	112	20112	1	0	0	21.17408	348	101	-0.10000
56	30112	41	40041	1	41	112	21.17408	351	101	-0.10000
57	10042	113	20113	1	0	0	18.51482	362	101	-0.10000
58	30113	42	40042	1	42	113	18.51482	365	101	-0.10000
59	10043	114	20114	1	0	0	15.85556	376	101	-0.10000
60	30114	43	40043	1	43	114	15.85556	379	101	-0.10000
61	10044	115	20115	1	0	0	13.19630	390	101	-0.10000
62	30115	44	40044	1	44	115	13.19630	393	101	-0.10000
63	10045	116	20116	-5	0	0	10.53704	404	101	-0.10000
64	30116	45	40045	-5	45	116	10.53704	407	101	-0.10000
65	10046	117	20117	1	0	0	7.87778	418	101	-0.10000
66	30117	46	40046	1	46	117	7.87778	421	101	-0.10000
67	10047	118	20118	1	0	0	5.21852	432	101	-0.10000
68	30118	47	40047	1	47	118	5.21852	435	101	-0.10000
69	10048	119	20119	1	0	0	2.55926	446	101	-0.10000
70	30119	48	40048	1	48	119	2.55926	449	101	-0.10000
71	10013	108	20108	-1	0	0	-0.0	457	101	-0.00050
72	30108	13	40013	-1	13	108	-0.0	451	101	-0.00050

SOLUTION FEASIBLE.

ITERATION TIME WAS 0.096 MINUTES

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BASIS INVERSION. BASIS SAVED.

ITERATION ACCOUNTING

INVERT TIME WAS 0.036 MINUTES.

SOLUTION FEASIBLE.

ITR	ENTER	BASIS	LEAVE	ZJ-CJ S	B	NB	INFEAS			
73	10049	49	40049	24	0	0	0.0	376	101	0.0
74	10050	50	40050	23	0	0	0.0	392	101	0.0
75	10051	51	40051	22	0	0	0.0	415	101	0.0
76	10052	52	40052	21	0	0	0.0	445	101	0.0
77	10053	53	40053	20	0	0	0.0	482	101	0.0
78	10054	54	40054	19	0	0	0.0	526	101	0.0
79	10055	55	40055	18	0	0	0.0	577	101	0.0
80	10056	56	40056	17	0	0	0.0	635	101	0.0
81	10057	57	40057	16	0	0	0.0	700	101	0.0
82	10058	58	40058	15	0	0	0.0	772	101	0.0
83	10059	59	40059	14	0	0	0.0	851	101	0.0
84	10060	60	40060	13	0	0	0.0	935	101	0.0
85	10025	25	40025	12	0	0	0.0	1019	101	0.0
86	10026	26	40026	11	0	0	0.0	1104	101	0.0
87	10027	27	40027	10	0	0	0.0	1190	101	0.0
88	10028	28	40028	9	0	0	0.0	1277	101	0.0
89	10029	51	10051	8	0	0	0.0	1365	101	0.0
90	40051	29	40029	8	29	51	0.0	1434	101	0.0
91	10030	52	10052	7	0	0	0.0	1504	101	0.0
92	40052	53	10053	7	30	52	0.0	1566	101	0.0
93	40053	30	40030	7	30	53	0.0	1620	101	0.0
94	10031	54	10054	6	0	0	0.0	1675	101	0.0
95	40054	31	40031	6	31	54	0.0	1722	101	0.0
96	10032	55	10055	5	0	0	0.0	1770	101	0.0
97	40055	32	40032	5	32	55	0.0	1810	101	0.0
98	10033	56	10056	4	0	0	0.0	1851	101	0.0
99	40056	33	40033	4	33	56	0.0	1884	101	0.0
100	10034	57	10057	3	0	0	0.0	1918	101	0.0
101	40057	58	10058	3	34	57	0.0	1944	101	0.0
102	40058	34	40034	3	34	58	0.0	1962	101	0.0
103	10035	59	10059	2	0	0	0.0	1981	101	0.0
104	40059	60	10060	2	35	59	0.0	1992	101	0.0
105	40060	35	40035	2	35	60	0.0	2001	101	0.0
106	10036	36	40036	1	0	0	0.0	2003	101	0.0

SOLUTION OPTIMAL.

AGENDUM TIME WAS 0.375 MINUTES

OUTPUT

OPTIMAL SOLUTION VALUES

STRUCTURAL VARR.		OPT. LEVEL
C101	10001	0.83987255D 03
C102	10002	0.77804647D 03
C103	10003	0.75000027D 03
C104	10004	0.75000027D 03
C105	10005	0.75000027D 03
C106	10006	0.75000027D 03
C107	10007	0.75000027D 03
C108	10008	0.75000027D 03
C109	10009	0.75000027D 03
C110	10010	0.75000027D 03
C111	10011	0.75000027D 03
C112	10012	0.75000027D 03
C113	10013	0.51185233D 04
C114	10014	0.37117985D 04
C115	10015	0.35992166D 04
C116	10016	0.35992166D 04
C117	10017	0.35992166D 04
C118	10018	0.35992166D 04
C119	10019	0.35992166D 04
C120	10020	0.35992166D 04
C121	10021	0.35992166D 04
C122	10022	0.35992166D 04
C123	10023	0.35992166D 04
C124	10024	0.35992166D 04
C125	10025	0.46067260D 04
C126	10026	0.47193079D 04
C127	10027	0.47193079D 04
C128	10028	0.47193079D 04
C129	10029	0.47193079D 04
C130	10030	0.47193079D 04
C131	10031	0.47193079D 04
C132	10032	0.47193079D 04
C133	10033	0.47193079D 04
C134	10034	0.47193079D 04
C135	10035	0.47193079D 04
C136	10036	0.31999976D 04
C137	10037	0.26592607D 02
C138	10038	0.26592607D 02
C139	10039	0.26592607D 02
C140	10040	0.26592607D 02
C141	10041	0.26592607D 02
C142	10042	0.26592607D 02
C143	10043	0.26592607D 02
C144	10044	0.26592607D 02
C145	10045	0.26592607D 02
C146	10046	0.26592607D 02
C147	10047	0.26592607D 02
C148	10048	0.26592607D 02
C149	10049	0.89872253D 01
C150	10050	0.28046192D 01

PRIMAL SLACKS / OPT. LEVEL

R185	20120	1	0.0
R186	20121	1	0.0
R187	20122	1	0.0
R188	20123	1	0.0
R189	20124	1	0.0
R190	20125	1	0.0
R191	20126	1	0.0
R192	20127	1	0.0
R193	20128	1	0.0
R194	20129	1	0.0
R195	20130	1	0.0
R196	20131	1	0.0

LAGRANGE MULT. / OPT. LEVEL

R149	30084	-1	0.0
R150	30085	-1	0.0
R151	30086	-1	-0.505615750D 02
R152	30087	-1	-0.339100480D 03
R153	30088	-1	-0.339100480D 03
R154	30089	-1	-0.339100480D 03
R155	30090	-1	-0.339100480D 03
R156	30091	-1	-0.339100480D 03
R157	30092	-1	-0.339100480D 03
R158	30093	-1	-0.339100480D 03
R159	30094	-1	-0.339100480D 03
R160	30095	-1	-0.339100480D 03
R161	30096	1	0.523663010D 02
R162	30097	-1	-0.185688190D 02
R163	30098	-1	-0.200549000D 02
R164	30099	-1	-0.200549000D 02
R165	30100	-1	-0.200549000D 02
R166	30101	-1	-0.200549000D 02
R167	30102	-1	-0.200549000D 02
R168	30103	-1	-0.200549000D 02
R169	30104	-1	-0.200549000D 02
R170	30105	-1	-0.200549000D 02
R171	30106	-1	-0.200549000D 02
R172	30107	-1	-0.200549000D 02
R173	30108	-1	-0.104730640D 05
R174	30109	-1	-0.371356510D 04
R175	30110	-1	-0.401078150D 04
R176	30111	-1	-0.401078150D 04
R177	30112	-1	-0.401078150D 04
R178	30113	-1	-0.401078150D 04
R179	30114	-1	-0.401078150D 04
R180	30115	-1	-0.401078150D 04
R181	30116	-1	-0.401078150D 04
R182	30117	-1	-0.401078150D 04
R183	30118	-1	-0.401078150D 04
R184	30119	-1	-0.401078150D 04

LAGRANGE SLACK / OPT. LEVEL

A103	40051	0.50561575D 02
A104	40052	0.33910048D 03
A105	40053	0.33910048D 03
A106	40054	0.33910048D 03
A107	40055	0.33910048D 03
A108	40056	0.33910048D 03
A109	40057	0.33910048D 03
A110	40058	0.33910048D 03
A111	40059	0.33910048D 03
A112	40060	0.33910048D 03
A113	40061	0.18568819D 02
A114	40062	0.20054900D 02
A115	40063	0.20054900D 02
A116	40064	0.20054900D 02
A117	40065	0.20054900D 02
A118	40066	0.20054900D 02
A119	40067	0.20054900D 02
A120	40068	0.20054900D 02
A121	40069	0.20054900D 02
A122	40070	0.20054900D 02
A123	40071	0.20054900D 02
A124	40072	0.10473064D 05
A125	40073	0.37135651D 04
A126	40074	0.40107815D 04
A127	40075	0.40107815D 04
A128	40076	0.40107815D 04
A129	40077	0.40107815D 04
A130	40078	0.40107815D 04
A131	40079	0.40107815D 04
A132	40080	0.40107815D 04
A133	40081	0.40107815D 04
A134	40082	0.40107815D 04
A135	40083	0.40107815D 04

FUNCTIONAL VALUE IS 0.99043118D 05

AGENDUM TIME WAS 0.074 MINUTES

CHECK

ROW	NAME	ORIGINAL RHS	SOLUTION VALUE	ROW ERROR
1	R101	-921.199463	-921.199463	-0.000000
2	R102	22.055969	22.055969	0.000000
3	R103	22.055969	22.055969	0.000000
4	R104	22.055969	22.055969	-0.000000
5	R105	22.055969	22.055969	0.000000
6	R106	22.055969	22.055969	0.000000
7	R107	22.055969	22.055969	0.000000
8	R108	22.055969	22.055969	-0.000000
9	R109	22.055969	22.055969	0.000000
10	R110	22.055969	22.055969	-0.000000
11	R111	22.055969	22.055969	0.000000
12	R112	22.055969	22.055969	0.000000
13	R113	4.095999	4.095999	0.000000
14	R114	4.095999	4.095999	0.000000
15	R115	4.095999	4.095999	0.000000
16	R116	4.095999	4.095999	0.000000
17	R117	4.095999	4.095999	0.000000
18	R118	4.095999	4.095999	-0.000000
19	R119	4.095999	4.095999	0.000000
20	R120	4.095999	4.095999	0.000000
21	R121	4.095999	4.095999	0.000000
22	R122	4.095999	4.095999	0.000000
23	R123	4.095999	4.095999	0.000000
24	R124	4.095999	4.095999	0.000000
25	R125	-4.223996	-4.223996	0.000000
26	R126	-4.223996	-4.223996	0.000000
27	R127	-4.223996	-4.223996	0.000000
28	R128	-4.223996	-4.223996	0.000000
29	R129	-4.223996	-4.223996	0.000000
30	R130	-4.223996	-4.223996	-0.000000
31	R131	-4.223996	-4.223996	0.000000
32	R132	-4.223996	-4.223996	-0.000000
33	R133	-4.223996	-4.223996	0.000000
34	R134	-4.223996	-4.223996	-0.000000
35	R135	-4.223996	-4.223996	-0.000000
36	R136	-4.223996	-4.223996	-0.000000
37	R137	-0.020000	-0.020000	0.000000
38	R138	-0.020000	-0.020000	-0.000000
39	R139	-0.020000	-0.020000	0.0
40	R140	-0.020000	-0.020000	-0.000000
41	R141	-0.020000	-0.020000	-0.000000
42	R142	-0.020000	-0.020000	0.000000
43	R143	-0.020000	-0.020000	-0.000000
44	R144	-0.020000	-0.020000	-0.000000
45	R145	-0.020000	-0.020000	-0.000000
46	R146	-0.020000	-0.020000	-0.000000
47	R147	-0.020000	-0.020000	-0.000000
48	R148	-0.020000	-0.020000	-0.000000
49	A101	0.0	0.0	0.0
50	A102	0.0	0.0	0.0
51	A103	0.0	0.000000	-0.000000
52	A104	0.0	0.000000	-0.000000
53	A105	0.0	0.000000	-0.000000
54	A106	0.0	0.000000	-0.000000
55	A107	0.0	0.000000	-0.000000
56	A108	0.0	0.0	0.0

57	A109	0.0	0.0	0.0
58	A110	0.0	0.000000	-0.000000
59	A111	0.0	0.000000	-0.000000
60	A112	0.0	0.000000	-0.000000
61	A113	0.0	0.0	0.0
62	A114	0.0	0.0	0.0
63	A115	0.0	0.0	0.0
64	A116	0.0	0.0	0.0
65	A117	0.0	0.0	0.0
66	A118	0.0	0.0	0.0
67	A119	0.0	0.0	0.0
68	A120	0.0	0.0	0.0
69	A121	0.0	0.0	0.0
70	A122	0.0	0.0	0.0
71	A123	0.0	0.0	0.0
72	A124	0.0	0.0	0.0
73	A125	0.0	0.0	0.0
74	A126	0.0	0.0	0.0
75	A127	0.0	0.0	0.0
76	A128	0.0	0.0	0.0
77	A129	0.0	0.0	0.0
78	A130	0.0	0.0	0.0
79	A131	0.0	0.0	0.0
80	A132	0.0	0.0	0.0
81	A133	0.0	0.0	0.0
82	A134	0.0	0.0	0.0
83	A135	0.0	0.0	0.0
84	R149	75.000000	75.000000	0.000000
85	R150	75.000000	75.000000	0.000000
86	R151	75.000000	75.000000	0.000000
87	R152	75.000000	75.000000	0.000000
88	R153	75.000000	75.000000	0.000000
89	R154	75.000000	75.000000	0.000000
90	R155	75.000000	75.000000	0.000000
91	R156	75.000000	75.000000	0.000000
92	R157	75.000000	75.000000	0.000000
93	R158	75.000000	75.000000	0.000000
94	R159	75.000000	75.000000	0.000000
95	R160	75.000000	75.000000	0.000000
96	R161	20.000000	20.000000	-0.000000
97	R162	300.000000	300.000000	0.000000
98	R163	300.000000	300.000000	0.000000
99	R164	300.000000	300.000000	0.000000
100	R165	300.000000	300.000000	0.000000
101	R166	300.000000	300.000000	0.000000
102	R167	300.000000	300.000000	0.000000
103	R168	300.000000	300.000000	0.000000
104	R169	300.000000	300.000000	0.000000
105	R170	300.000000	300.000000	0.000000
106	R171	300.000000	300.000000	0.000000
107	R172	300.000000	300.000000	0.000000
108	R173	2.659260	2.659260	0.000000
109	R174	2.659260	2.659260	0.000000
110	R175	2.659260	2.659260	0.000000
111	R176	2.659260	2.659260	0.000000
112	R177	2.659260	2.659260	0.000000
113	R178	2.659260	2.659260	0.000000
114	R179	2.659260	2.659260	0.000000
115	R180	2.659260	2.659260	0.000000
116	R181	2.659260	2.659260	0.000000
117	R182	2.659260	2.659260	0.000000

118	R183	2.659260	2.659260	0.000000
119	R184	2.659260	2.659260	0.000000
120	R185	2.659260	2.659260	0.000000
121	R186	2.659260	2.659260	0.000000
122	R187	2.659260	2.659260	0.000000
123	R188	2.659260	2.659260	0.000000
124	R189	2.659260	2.659260	0.000000
125	R190	2.659260	2.659260	0.000000
126	R191	2.659260	2.659260	0.000000
127	R192	2.659260	2.659260	0.000000
128	R193	2.659260	2.659260	0.000000
129	R194	2.659260	2.659260	0.000000
130	R195	2.659260	2.659260	0.000000
131	R196	2.659260	2.659260	0.000000

MAXIMUM ERROR IN ROW 47 0.125055520-11

AGENDUM TIME WAS 0.035 MINUTES

```
        INVERT              PRICING FRACTION  1.000
                  ITERATION ACCOUNTING
  ITR     ENTER  BASIS   LEAVE     ZJ-CJ S     B    NB      INFEAS

                                                          INVERT TIME WAS   0.036 MINUTES.

SOLUTION FEASIBLE.
..

SOLUTION OPTIMAL.

AGENDUM TIME WAS   0.173 MINUTES
```

```
*******************************************************************************************************************************

FORTRAN IV G LEVEL 1, MOD 4            MAIN              DATE = 69283          16/56/28                  PAGE 0001

                 C        VALUE OF THE OBJECTIVE FUNCTION
      0001                DOUBLE PRECISION A(48,48),B(48), X(128),V1,V2,V3,DV1,DV2,TVAL
      0002                DATA V1,V2,V3/0.,0.,0./,X/128*0./
      0003                READ(19,1)  IND, NTIME,ICOND,SCALE
      0004              1 FORMAT(3I3,F12.4)
      0005                READ(19,20)                              C1,C2,C3,C4,C5,C6,C7,C8,C9,PAR1,
                         1W1,W2,WOR1, XINVI,WORK,UPPY,YLOW,PAR2
      0006             20 FORMAT((6E12.4))
      0007                L3 = 3*NTIME
      0008                L4 = 4*NTIME
      0009                READ(19,2)  ((A(I,J), J=1,L4), I=1,L4),  (B(I), I=1,L4)
      0010              2 FORMAT(4X,E015.8)
      0011                READ(20,3,END=4) (IX,X(IX), J=1,128)
      0012              3 FORMAT(4X,I2,19X,D15.8)
      0013              4 WRITE(3,60) NTIME, W1, W2, C1,C2,C3,C4,C5,C6,C7,C8,C9,WOR1,WORK,
                         1XINVI, PAR1,PAR2,     UPPY, YLOW,SCALE
      0014             60 FORMAT('1',50X,'THE PROBLEM PARAMETER ARE'/'0',30X,'NTIME = ',  I5
                         1 ,10X,'W1     = ',F14.8,10X,'W2     = ',F14.8/'0',30X,'C1     = ',E14
                         2.8,10X,'C2     = ',F14.8,10X,'C3     = ',E14.8/'0',30X,'C4     = ',F1
                         34.8,10X,'C5     = ',E14.8,10X,'C6     = ',F14.8/'0',30X,'C7     = ',F
                         414.8,10X,'C8     = ',F14.8,'C9     = ',F14.8/'0',30X,'INT WOR = ',F1
                         54.8,8X,'WOR RES = ',F14.8,8X,'INT INV = ',F14.8/'0',40X,'AVER SALE
                         6 = ',F14.8,6X,'MIN SALE  = ',F14.8/'0'40X,'UPPER Y = ',F14.8,15X,'
                         7LOWER Y = ',F14.8/'0',60X,'SCALE = ',E15.8////)
      0015                DO 5  K=1,L4
      0016                X(K) = X(K)/SCALE
      0017                DO 5 J =1,L4
      0018              5 A(I,J) = +A(I,J)*.5
      0019                DO 6  I=1,L3
      0020                V2 = V2 - X(I)*B(I)
      0021                DO 6 J =1,L3
      0022              6 V1 = V1 + X(I)*X(J)*A(I,J)
      0023                IB = L3 +1
      0024                DO 7  I =IB,L4
      0025              7 V3 = V3 - X(I)*B(I)
      0026                DV1 = -V1/W1
      0027                DV2 = -V2/W1
      0028                TVAL =  DV1 + DV2
      0029                DV3 = -V3/W2
                 C                    EVALUATION OF CONSTANT TERM
      0030                CC = 24.*C7*C8*C9*PAR1  + 12.*C7*C8**2 + C2*WOR1
      0031                COST=TVAL+CC
      0032                WRITE(3,9)  ( I,X(I), X(I+NTIME),X(I+2*NTIME),X(I+L3),I=1,NTIME)
      0033              9 FORMAT('0',////60X,'THE  RESULTS'////'0',15X,'PERIOD',8X,'WORK LEV
                         1EL',10X,'PRODUCTION LEVEL',7X,'INVENTORY LEVEL',7X,'RELIABILITY LE
                         2VEL'///('0',17X,I2,4(8X,D15.8)))
      0034                WRITE(3,10) COST,CC,DV1
      0035                WRITE(3,11) DV2,TVAL,DV3
      0036             10 FORMAT('0',                       ////'0', 5X,'TOTAL COST = ',D15.8,
                         35X,'FIXED COST = ',D15.8, 5X,'QUADRATIC COST = ',D15.8)
      0037             11 FORMAT('0',                                                  17X,'LINEAR
                         4 COST = ' ,D15.8,5X,'VARIABLE COST = ',D15.8,      5X,'RELIABILITY
                         5COST = ',D15.8///'0',50X,'--------------------------------------')
```

THE PROBLEM PARAMETER ARE

NTIME = 12 W1 = 0.79999995E 00 W2 = 0.19999999E 00

C1 = 0.34000000E 03 C2 = 0.64299988E 02 C3 = 0.19999999E 00

C4 = 0.56699991E 01 C5 = 0.51199997E 02 C6 = 0.28100000E 03

C7 = 0.82499981E-01 C8 = 0.32000000E 03C9 = 0.0

INT WOR = 0.90000000E 02 WOR RES = 0.75000000E 02 INT INV = 0.32000000E 03

AVER SALE = 0.50000000E 03 MIN SALE = 0.30000000E 03

UPPER Y = 0.26589994E 01 LOWER Y = 0.26589994E 01

SCALE = 0.10000000E 02

THE RESULTS

PERIOD	WORK LEVEL	PRODUCTION LEVEL	INVENTORY LEVEL	RELIABILITY LEVEL
1	0.83987255D 02	0.51185233D 03	0.46067260D 03	0.26592607D 01
2	0.77804647D 02	0.37117985D 03	0.47193079D 03	0.26592607D 01
3	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
4	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
5	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
6	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
7	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
8	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
9	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
10	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
11	0.75000027D 02	0.35992166D 03	0.47193079D 03	0.26592607D 01
12	0.75000027D 02	0.35992166D 03	0.31999976D 03	0.26592607D 01

```
TOTAL COST =  0.65934750E 06     FIXED COST =  0.10716287E 06     QUADRATIC COST =  0.13519795D 07
          LINEAR COST = -0.79979482D 06     VARIABLE COST =  0.55218463D 06     RELIABILITY COST = -0.31911118E 02

                                   -----------------------------------

SUMMARY OF ERRORS FOR THIS JOB   ERROR NUMBER   NUMBER OF ERRORS
                                     217               1
```

10. APPENDIX B: A NONLINEAR PROGRAMMING SOLUTION

Our problem is

$$\text{Min } f(\bar{x}) \equiv -.05(x_1+x_2+x_3)-\ln x_4 - \ln x_5 - \ln x_6$$

$$- \ln x_7 - \ln x_8$$

subject to

$$\begin{aligned}
g_1(\bar{x}) &\equiv 1300 + (.05-.15x_4)x_1 && \geq 0 \\
g_2(\bar{x}) &\equiv 1000 + (.05-.15x_5)x_2 && \geq 0 \\
g_3(\bar{x}) &\equiv 1000 + (.05-.15x_6)x_3 && \geq 0 \\
g_4(\bar{x}) &\equiv 7000 \qquad\qquad -x_1 && \geq 0 \\
g_5(\bar{x}) &\equiv 5500 + (.05-.15x_7)x_1 && \geq 0 \\
g_6(\bar{x}) &\equiv -.15x_8\sqrt{x_1^2+x_2^2} + 9000 + .05x_1 + .05x_2-x_3 && \geq 0 \\
g_7(\bar{x}) &\equiv x_4x_5x_6x_7x_8-1.29^5 && \geq 0 \\
g_8(\bar{x}) &\equiv 4.-x_4 && \geq 0 \\
g_9(\bar{x}) &\equiv 4.-x_5 && \geq 0 \\
g_{10}(\bar{x}) &\equiv 4.-x_6 && \geq 0 \\
g_{11}(\bar{x}) &\quad 4.-x_7 && \geq 0 \\
g_{12}(\bar{x}) &\quad 4.-x_8 && \geq 0
\end{aligned}$$

The program AUKLET (126) modified by the author uses sequential unconstrained minimization technique (SUMT) developed by V. Fiacco and G. P. McCormick (82).

Three additional subroutines are needed before the program is executable. These three FORTRAN IV subroutines are written by the user and are:

RESTNT defines the values of $f(\bar{x})$ and $g_i(\bar{x})$

GRAD1 defines the first derivates of $f(\bar{x})$ and $f_i(\bar{x})$

MATRIX defines the partial derivates of $f(\bar{x})$ and $g_i(\bar{x})$

So we must supply the following system of equations

For RESTNT subroutine

$$f(\bar{x}) = -.05(x_1+x_2+x_3) - \sum_{k=4}^{8} \ln x_k$$

$$g_1(\bar{x}) = 1000 + (.05-.15x_4)x_1$$

$$g_2(\bar{x}) = 1000 + (.05-.15x_5)x_2$$

$$g_3(\bar{x}) = 1000 + (.05-.15x_6)x_3$$

$$g_4(\bar{x}) = 7000 - x_1$$

$$g_5(\bar{x}) = 5500 + (.05-.15x_7)x_1-x_2$$

$$g_6(\bar{x}) = -.15(x_8\sqrt{x_1^2+x_2^2} + 9000 + .05x_1 + .05x_2 - x_3$$

$$g_7(\bar{x}) = x_4x_5x_6x_7x_8 - (1.29)^5$$

$$g_8(\bar{x}) = 4-x_4$$

$$g_9(\bar{x}) = 4-x_5$$

$$g_{10}(\bar{x}) = 4-x_6$$

$$g_{11}(\bar{x}) = 4-x_7$$

$$g_{12}(\bar{x}) = 4-x_8$$

For GRAD1 subroutine

We evaluate the first derivates

For $f(\bar{x})$

$$\frac{\partial f(\bar{x})}{\partial x_i} = -.05 \qquad i=1,2,3$$

$$\frac{\partial f(\bar{x})}{\partial x_i} = -\frac{1}{x_i} \qquad i=4,5,6,7,8$$

For $g_1(\bar{x})$

$$\frac{\partial g_1(\bar{x})}{\partial x_1} = .05-.15x_4 \quad \frac{\partial g_1(\bar{x})}{\partial x_4} = -.15x_1 \quad \frac{\partial g_1(\bar{x})}{\partial x_k} = 0$$

for $k \neq 1$ or 4

For $g_2(\bar{x})$

$$\frac{\partial g_2(\bar{x})}{\partial x_2} = .05-.15x_4 \quad \frac{\partial g_2}{\partial x_5} = .15x_2 \quad \frac{\partial g_2(\bar{x})}{\partial x_k} = 0$$

for $k \neq 2$ or 5

For $g_3(\bar{x})$

$$\frac{\partial g_3(\bar{x})}{\partial x_3} = .05-.15x_6 \quad \frac{\partial g_3(\bar{x})}{\partial x_6} = -.15x_3 \quad \frac{\partial g_3(\bar{x})}{\partial x_k} = 0$$

for any $k \neq 3$ or 6

For $g_4(\bar{x})$

$$\frac{\partial g_4(\bar{x})}{\partial x_1} = -1 \qquad \frac{\partial g_4(\bar{x})}{\partial x_k} = 0 \quad \text{for any } k \neq 1$$

For $g_5(\bar{x})$

$$\frac{\partial g_5(\bar{x})}{\partial x_7} = .05 - .15x_7 \qquad \frac{\partial g_5(\bar{x})}{\partial x_2} = -1 \qquad \frac{\partial g_5(\bar{x})}{\partial\ x_1} = -.15x_1$$

$$\frac{\partial g_5(\bar{x})}{\partial x_k} = 0 \quad \text{for any } k \neq 1,\ 2,\ \text{or } 7$$

For $g_6(\bar{x})$

$$\frac{\partial g_6(\bar{x})}{\partial x_1} = \frac{-.05x_8x_1}{\sqrt{x_1^2+x_2^2}} + .05 \qquad \frac{\partial g_6(\bar{x})}{\partial x_2} = \frac{-.05x_8x_2}{\sqrt{x_1^2+x_2^2}} + .05$$

$$\frac{\partial g_6(\bar{x})}{\partial x_3} = -1 \qquad \frac{\partial g_6(\bar{x})}{\partial x_8} = -.05\sqrt{x_1^2+x_2^2} \qquad \frac{\partial g_6(\bar{x})}{\partial x_k} = 0$$

for $k = 4,5,6,7$

For $g_7(\bar{x})$

$$\frac{\partial g_7(\bar{x})}{\partial x_k} = 0 \quad \text{for } k = 1,2,3$$

$$\frac{\partial g_7(\bar{x})}{\partial x_k} = \frac{x_4x_5x_6x_7x_8}{x_k} \quad \text{for } k=4,5,6,7,8$$

For $g_j(\bar{x})$ $j=8,9,10,11,12$

$$\frac{\partial g_j(\bar{x})}{\partial x_k} = -1. \quad \text{for } j=k;\ k=4,5,6,7,8$$

For MATRIX subroutine

We evaluate the partial derivates of $f(\bar{x})$ and $g_i(x)$.

For $f(\bar{x})$

$$\frac{\partial^2 f(\bar{x})}{\partial x_k^2} = 1/x_k^2 \qquad k=4,5,6,7,8$$

$$\frac{\partial^2 f(\bar{x})}{\partial x_i^2} = 0 \qquad i=1,2,3$$

$$\frac{\partial^2 f(\bar{x})}{\partial x_i \partial x_j} = 0 \qquad i=1,\ldots,8 \quad j=1,\ldots,8$$

For $g_1(\bar{x})$

$$\frac{\partial^2 g_1(\bar{x})}{\partial x_i \partial x_j} = 0 \quad i=j;\ i,j\neq 1,4;\ i=1,\ldots,8;\ j=1,\ldots,8$$

$$\frac{\partial^2 g_1(\bar{x})}{\partial x_1 \partial x_4} = \frac{\partial^2 g_1(\bar{x})}{\partial x_4 \partial x_1} = -.15$$

For $g_2(\bar{x})$

$$\frac{\partial^2 g_2(\bar{x})}{\partial x_i \partial x_j} = 0 \quad i=j;\ i,j\neq 2,5;\ i=1,\ldots 8;\ j=1,\ldots,8$$

$$\frac{\partial^2 g_2(\bar{x})}{\partial x_2 \partial x_5} = \frac{\partial^2 g_2(\bar{x})}{\partial x_5 \partial x_2} = -.15$$

For $g_3(\bar{x})$

$$\frac{\partial^2 g_3(\bar{x})}{\partial x_i \partial x_j} = 0 \quad i=j;\ i,j\neq 3,6;\ i=1,\ldots,8;\ j=1,\ldots,8$$

$$\frac{\partial^2 g_3(\bar{x})}{\partial x_3 \partial x_6} = \frac{\partial^2 g_3(\bar{x})}{\partial x_6 \partial x_3} = -.15$$

For $g_4(\bar{x})$

$$\frac{\partial^2 g_4(\bar{x})}{\partial x_i^2} = 0 \quad \forall_i \quad \frac{\partial^2 g_5(\bar{x})}{\partial x_i \partial x_j} = 0 \qquad \forall_{i,j}$$

For $g_5(\bar{x})$

$$\frac{\partial^2 g_5(\bar{x})}{\partial x_i \partial x_j} = 0 \quad i=j;\ i,j\neq 1,7;\ i=1,\ldots,8;\ j=1,\ldots,8$$

$$\frac{\partial^2 g_5(\bar{x})}{\partial x_1 \partial x_7} = \frac{\partial^2 g_5(\bar{x})}{\partial x_7 \partial x_1} = -.15$$

For $g_6(\bar{x})$

$$\frac{\partial^2 g_6(\bar{x})}{\partial x_1^2} = -.15 x_2^2 x_8 / (x_1^2+x_2^2)^{\frac{3}{2}}$$

$$\frac{\partial^2 g_6(\bar{x})}{\partial x_1 \partial x_2} = \frac{\partial^2 g_6(\bar{x})}{\partial x_2 \partial x_1} = .15 x_1 x_2 x_8 / (x_1^2+x_2^2)^{\frac{3}{2}}$$

$$\frac{\partial^2 g_6(\bar{x})}{\partial x_1 \partial x_8} = \frac{\partial^2 g_6(\bar{x})}{\partial x_8 \partial x_1} = -.15x_1/(x_1^2+x_2^2)^{\frac{1}{2}}$$

$$\frac{\partial^2 g_6(\bar{x})}{\partial x_2^2} = -.15x_1^2 x_8/(x_1^2+x_2^2)^{\frac{3}{2}}$$

$$\frac{\partial^2 g_6(\bar{x})}{\partial x_2 \partial x_8} = \frac{\partial^2 g_6(\bar{x})}{\partial x_8 \partial x_2} = -.15x_2/(x_1^2+x_2^2)^{\frac{1}{2}}$$

Any other partial derivative of $g_6(\bar{x})$ does not indicate above has value equal to zero.

For $g_7(\bar{x})$

$$\frac{\partial^2 g_7(\bar{x})}{\partial x_i \partial x_j} = x_4 x_5 x_6 x_7 x_8 / x_i x_j \quad i \neq j \quad i=4,\ldots,8; \; j=4,\ldots,8$$

Any other partial derivative of $g_7(\bar{x})$ does not indicated here has value equal to zero.

For $g_h(\bar{x})$ $\quad h=8,9,10,11,12$

All partial derivatives have value equal to zero because they are linear constraints.

Follows a computer point out of FORTRAN subroutines RESNT, GRAD1 MATRIX and the actual output.

```
FORTRAN IV G LEVEL 1, MOD 4            MAIN            DATE = 69258         23/24/13            PAGE 0001

      C     SUBROUTINES FOR RELIABILITY NASLUND PROBLEM
            SUBROUTINE RESTNT(IN,VAL)
            IMPLICIT REAL*8(A-H,O-Z)
            COMMON/SHARE/X(100),DEL(100), A(100,100),N,M,MN,NP1,NM1
            J = IN +1
            GO TO(1,2,3,4,5,6,7,8,9,10, 11,12,13), J
          1 VAL =-.05*(X(1)+X(2)+X(3))-DLOG(X(4))-DLOG(X(5))-DLOG(X(6))-DLOG(X
           1(7))-DLOG(X(8))
            GO TO 14
          2 VAL = 1300. + (.05-.15*X(4))*X(1)
            GO TO 14
          3 VAL = 1000. + (.05 - .15*X(5))*X(2)
            GO TO 14
          4 VAL = 1000. + (.05 - .15*X(6))*X(3)
            GO TO 14
          5 VAL = 7000. - X(1)
            GO TO 14
          6 VAL = 5500.+(.05-.15*X(7))*X(1) -X(2)
            GO TO 14
          7 VAL=-.15*X(8)*DSQRT(X(1)**2+X(2)**2)+9000.+.05*(X(1)+X(2)) -X(3)
            GO TO 14
          8 VAL = X(4)*X(5)*X(6)*X(7)*X(8) - (1.29)**5
            GO TO 14
          9 VAL = 4.  - X(4)
            GO TO 14
         10 VAL = 4.  - X(5)
            GO TO 14
         11 VAL = 4.  - X(6)
            GO TO 14
         12 VAL = 4.  - X(7)
            GO TO 14
         13 VAL = 4.  - X(8)
         14 RETURN
            END
```

```
      SUBROUTINE GRADI(I)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SHARE/X(100),DEL(100), A(100,100),N,M,MN,NP1,NM1
      DO 101 K=1,N
  101 DEL(K)=0.
      J = I + 1
      GO TO(1,2,3,4,5,6,7,8,9,10,11,12,13),  J
    1 DO 19 K=1,3
   19 DEL(K)=-.05
      DO 20 K=4,8
   20 DEL(K)= -1./X(K)
      GO TO 14
    2 DEL(1) = .05-.15*X(4)
      DEL(4)=-.15*X(1)
      GO TO 14
    3 DEL(2) = .05-.15*X(5)
      DEL(5)=-.15*X(2)
      GO TO 14
    4 DEL(3) = .05-.15*X(6)
      DEL(6)=-.15*X(3)
      GO TO 14
    5 DEL(1)=-1
      GO TO 14
    6 DEL(1) = .05-.15*X(7)
      DEL(2)= -1.
      DEL(7)=-.15*X(1)
      GO TO 14
    7 Q =DSQRT(X(1)**2 +X(2)**2)
      DEL(1)= -.15*X(1)*X(8)/Q + .05
      DEL(2)= -.15*X(2)*X(8)/Q + .05
      DEL(3)= -1.
      DEL(8)= -.15*Q
      GO TO 14
    8 DO 80 K=4,8
   80 DEL(K)= X(4)*X(5)*X(6)*X(7)*X(8)/X(K)
      GO TO 14
    9 DEL(4) =-1.
      GO TO 14
   10 DEL(5) =-1.
      GO TO 14
   11 DEL(6) =-1.
      GO TO 14
   12 DEL(7) =-1.
      GO TO 14
   13 DEL(8) =-1.
   14 RETURN
      END
```

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FORTRAN IV G LEVEL 1, MOD 4 MATRIX DATE = 69258 23/24/13 PAGE 0001

```
      SUBROUTINE MATRIX(I)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/SHARE/X(100),DEL(100), A(100,100),N,M,MN,NP1,NM1
      J = I +1
      DO 100 K=1,N
      DO 100 L=1,N
  100 A(K,L) = 0.
      GO TO(1,2,3,4,5,6,7,8,9,9,9,9,9), J
    1 DO 11 L=4,6
   11 A(L,L)= 1./(X(L))**2
      GO TO 9
    2 A(1,4)=-.15
      GO TO 9
    3 A(2,5)=-.15
      GO TO 9
    4 A(3,6) = -.15
      GO TO 9
    5 A(1,7)=-.15
      GO TO 9
    7 B =DSQRT(X(1)**2 +X(2)**2)
      B3 = B**3
      A(1,1) = -.15*X(8)*(X(2)**2)/B3
      A(1,2) =  .15*X(1)*X(2)*X(8)/B3
      A(1,8) = -.15*X(1)/B
      A(2,2) = -.15*X(8)*(X(1)**2)/B3
      A(2,8) = -.15*X(2)/B
      GO TO 9
    8 DO 80 K=4,8
      DO 80 L=K,8
      IF(K.NE.L)  A(K,L)=X(4)*X(5)*X(6)*X(7)*X(8)/(X(K)*X(L))
   80 CONTINUE
    9 RETURN
      END
```

```
    1       1       1       1       1       1       1       1       1       1       0
 0.1000D-04  0.1000D 03  0.1000D-03  0.4000D 02  0.0          12    6    0
 0.5900D 04  0.3000D 04  0.5000D 04  0.1650D 01  0.1650D 01  0.1650D 01
 0.1650D 01  0.1650D 01

 OPTIONS SELECTED

    N=  8        M= 12        MZ=  0

      MAX. TIME= 0.0                K= 1.0000000D 02     RATIO= 4.0000000D 01       EPSILON= 1.0000000D-05     THETA= 1.0000000D-04
 NONLINEAR PROGRAMMING ROUTINE-PAC

       F= -6.9750354D 02       P=  0.0                 G=  0.0                 RSIGMA=  0.0                H=  0.0

     THE CURRENT VALUE OF X IS
     5.9000000D 03        3.0000000D 03        5.0000000D 03        1.6500000D 00        1.6500000D 00        1.6500000D 00
     1.6500000D 00        1.6500000D 00
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      1.3475021D 02        4.0750011D 02        1.2500182D 01        1.1000000D 03        1.3347502D 03        2.8068194D 03
      3.6575035D 03        2.3500000D 00        2.3500000D 00        2.3500000D 00        2.3500000D 00        2.3500000D 00
0******THE FEASIBLE STARTING POINT TO BE USED IS ...

       F= -6.9750354D 02       P=  0.0                 G=  0.0                 RSIGMA=  0.0                H=  0.0

     THE CURRENT VALUE OF X IS
     5.9000000D 03        3.0000000D 03        5.0000000D 03        1.6500000D 00        1.6500000D 00        1.6500000D 00
     1.6500000D 00        1.6500000D 00
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      1.3475021D 02        4.0750011D 02        1.2500182D 01        1.1000000D 03        1.3347502D 03        2.8068194D 03
      3.6575035D 03        2.3500000D 00        2.3500000D 00        2.3500000D 00        2.3500000D 00        2.3500000D 00
 APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

 ******************************************************

           POINT=  33        DOTT=  6.8520804D-04      RHO=  1.9859739D 03      MAGNITUDE=  1.1625088D 00      PHASE= 2

       F= -6.3348542D 02       P=  9.4503570D 03       G= -1.0718128D 04       RSIGMA=  1.0084242D 04      H=  0.0

     THE CURRENT VALUE OF X IS
     5.1464835D 03        3.6859677D 03        3.7775463D 03        1.9895219D 00        1.9487911D 00        1.9268511D 00
     2.0373106D 00        2.0432325D 00
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      3.8665379D 01        1.0682141D 02        9.7062056D 01        1.8535165D 03        4.9860895D 02        3.7239365D 03
      2.5962932D 01        2.1104761D 00        2.0512089D 00        2.0731489D 00        1.9626894D 00        1.9567675D 00


 LAGRANGE MULTIPLIERS

       F= -6.3348542D 02       P=  9.4503570D 03       G= -1.0718128D 04       RSIGMA=  1.0084242D 04      H=  0.0

     THE CURRENT VALUE OF X IS
     7.4951197D-05        1.4617412D-04        1.3917266D-04        5.5624954D 02        5.2292930D 02        5.3490568D 02
     4.7847475D 02        4.7570528D 02
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      2.0403645D-01        1.7404313D-01        2.1080192D-01        5.7807026D-04        7.9882823D-03        1.4320857D-04
      2.9462262D 02        4.4587369D 02        4.7201276D 02        4.6207510D 02        5.1554951D 02        5.1867471D 02
```

```
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

***************************************************

       POINT= 387       DOTT=  1.0101066D-04      RHO=  4.9649348D 01      MAGNITUDE=  4.4290387D-01      PHASE= 2

     F= -9.5837030D 02      P= -6.6030639D 02      G= -1.2564342D 03      RSIGMA=  2.9806391D 02      H=  0.0

   THE CURRENT VALUE OF X IS
   6.9602250D 03       4.2457436D 03        7.9241784D 03        1.5432970D 00        1.8480071D 00        1.1494924D 00
   1.5002834D 00       1.3099672D 00
   THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
    3.6757356D 01       3.5362703D 01        2.9891641D 01        3.9774986D 01        3.5921363D 01        3.4099943D 01
    2.8707700D 00       2.4567030D 00        2.1519929D 00        2.8505076D 00        2.4997166D 00        2.6900328D 00


 1ST  ORDER ESTIMATES

     F= -1.0192276D 03      P= -2.5591811D 03      G= -1.0138267D 03      RSIGMA=  0.0                H=  0.0

   THE CURRENT VALUE OF X IS
   7.3008622D 03       4.3508746D 03        8.7029537D 03        1.4782728D 00        1.8290790D 00        1.0034974D 00
   1.3994248D 00       1.1722533D 00
   THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
    4.6143474D 01       2.3829924D 01        1.2513917D 02       -3.0086219D 02       -1.8382444D 01       -6.1481062D 02
    8.7886934D-01       2.5217272D 00        2.1709210D 00        2.9965026D 00        2.6005752D 00        2.8277467D 00


LAGRANGE MULTIPLIERS

     F= -1.0192276D 03      P= -2.5591811D 03      G= -1.0138267D 03      RSIGMA=  0.0                H=  0.0

   THE CURRENT VALUE OF X IS
   1.0248658D-06       2.7542669D-06        7.9068783D-07        2.0845607D 01        1.4538054D 01        3.7575201D 01
   2.2058041D 01       2.8932950D 01
   THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
    3.6747265D-02       3.9702938D-02        5.5566626D-02        3.1382929D-02        3.8477596D-02        4.2697874D-02
    6.0244408D 00       8.2263697D 00        1.0720916D 01        6.1103931D 00        7.9456970D 00        6.8611733D 00
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

***************************************************

       POINT=1630       DOTT=  1.5761148D-04      RHO=  1.2412337D 00      MAGNITUDE=  1.4343829D 00      PHASE= 2

     F= -9.9516265D 02      P= -9.8257174D 02      G= -1.0077536D 03      RSIGMA=  1.2590911D 01      H=  0.0

   THE CURRENT VALUE OF X IS
   6.9940514D 03       4.5133009D 03        8.3687957D 03        1.5665527D 00        1.8024903D 00        1.1253714D 00
   1.2684364D 00       9.6210441D-01
   THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
    6.2203031D 00       5.3880325D 00        5.7394406D 00        5.9486242D 00        5.6754961D 00        5.3087399D 00
    3.0567066D-01       2.4334473D 00        2.1975097D 00        2.8746286D 00        2.7315636D 00        3.0378956D 00


 2ND  ORDER ESTIMATES

     F= -1.0016149D 03      P= -1.0042221D 03      G= -1.0013771D 03      RSIGMA=  0.0                H=  0.0
```

```
  THE CURRENT VALUE OF X IS
  6.9927032D 03        4.5690038D 03        8.4458718D 03        1.5732959D 00        1.7930409D 00        1.1238501D 00
  1.2254152D 00        8.8970901D-01
  THE CONSTRAINT VALUES
                             NOT INCLUDING THE NON-NEGATIVITIES
  -6.0274827D-01       -4.1125751D-01       -1.4903042D 00        7.2997744D 00        5.2856077D-01        1.7444261D 01
  -1.2956748D-01        2.4267041D 00        2.2069591D 00        2.8761499D 00        2.7795818D 00        3.1102910D 00

 1ST  ORDER ESTIMATES

   F= -1.0020557D 03     P= -1.0430961D 03     G= -1.0013771D 03     RSIGMA=  0.0               H=  0.0

  THE CURRENT VALUE OF X IS
  7.0006043D 03        4.5635506D 03        8.4522989D 03        1.5709203D 00        1.7939419D 00        1.1208413D 00
  1.2248934D 00        8.9677261D-01
  THE CONSTRAINT VALUES
                             NOT INCLUDING THE NON-NEGATIVITIES
   4.5884311D-01        1.6603044D-01        1.5623065D 00       -4.0427479D-01        2.5747543D-01        1.8124190D 00
  -1.0263906D-01        2.4290797D 00        2.2060581D 00        2.8791587D 00        2.7751066D 00        3.1032274D 00

LAGRANGE MULTIPLIERS

   F= -1.0020557D 03     P= -1.0430961D 03     G= -1.0013771D 03     RSIGMA=  0.0               H=  0.0

  THE CURRENT VALUE OF X IS
  2.5374408D-08        6.0934742D-08        1.7722608D-08        5.0578220D-01        3.8203897D-01        9.8008058D-01
  7.7146494D-01        1.3409394D 00
  THE CONSTRAINT VALUES
                             NOT INCLUDING THE NON-NEGATIVITIES
   3.2079712D-02        4.2755614D-02        3.7680307D-02        3.5076843D-02        3.8534160D-02        4.4042368D-02
   1.3254526D 01        2.0960887D-01        2.5703483D-01        1.5020697D-01        1.6635310D-01        1.3449554D-01
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

*************************************************

     POINT=1742         DOTT=  1.2189902D-04       RHO=  3.1030842D-02       MAGNITUDE=  3.9235538D 00       PHASE= 2

   F= -9.9983882D 02     P= -9.9880014D 02     G= -1.0008775D 03     RSIGMA=  1.0386786D 00     H=  0.0

  THE CURRENT VALUE OF X IS
  6.9990663D 03        4.5576718D 03        8.4143112D 03        1.5706324D 00        1.7947904D 00        1.1248929D 00
  1.2300383D 00        9.2803524D-01
  THE CONSTRAINT VALUES
                             NOT INCLUDING THE NON-NEGATIVITIES
   1.0095618D 00        8.7391671D-01        9.3593329D-01        9.3369257D-01        9.1382567D-01        8.5496365D-01
   4.7481229D-02        2.4293676D 00        2.2052096D 00        2.8751071D 00        2.7699617D 00        3.0719648D 00

 2ND  ORDER ESTIMATES

   F= -1.0006824D 03     P= -1.0007903D 03     G= -1.0007012D 03     RSIGMA=  0.0               H=  0.0

  THE CURRENT VALUE OF X IS
  6.9999980D 03        4.5660679D 03        8.4221046D 03        1.5714108D 00        1.7933290D 00        1.1249046D 00
  1.2227736D 00        9.2227428D-01
  THE CONSTRAINT VALUES
                             NOT INCLUDING THE NON-NEGATIVITIES
   1.9263737D-02        3.4268639D-02       -4.1942260D-03        1.9995841D-03        2.0045292D-02        3.2942292D-03
   2.6616219D-03        2.4285892D 00        2.2066710D 00        2.8750954D 00        2.7772262D 00        3.0777257D 00
```

```
   1ST  ORDER ESTIMATES

      F= -1.0007158D 03     P= -1.0018480D 03     G= -1.0007012D 03     RSIGMA=  0.0              H=  0.0

     THE CURRENT VALUE OF X IS
     7.0000082D 03       4.5660050D 03       8.4228595D C3       1.5713985D 00       1.7933443D 00       1.1248030D 00
     1.2228268D 00       9.2163673D-01
     THE CONSTRAINT VALUES
                                  NOT INCLUDING THE NON-NEGATIVITIES
      3.0252493D-02       3.7557043D-02       3.4519712D-02      -8.1572752D-03       2.5984194D-02       4.8619921D-02
      2.5193290D-05       2.4266015D 00       2.2066557D 00       2.8751970D 00       2.7771732D 00       3.0783633D 00

  LAGRANGE MULTIPLIERS

      F= -1.0007168D 03     P= -1.0018480D 03     G= -1.0007012D 03     RSIGMA=  0.0              H=  0.0

     THE CURRENT VALUE OF X IS
     6.3345147D-10       1.4938517D-09       4.3828481D-10       1.2578953D-02       9.6331001D-03       2.4522866D-02
     2.0509559D-02       3.6030031D-02
     THE CONSTRAINT VALUES
                                  NOT INCLUDING THE NON-NEGATIVITIES
      3.0445825D-02       4.0630623D-02       3.5424503D-02       3.5594735D-02       3.7159243D-02       4.2452011D-02
      1.3764158D C1       5.2578365D-03       6.3810747D-03       3.7539243D-03       4.0443248D-03       3.2882218D-03
 APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

****************************************************

        POINT=1844       DOTT=  1.2102176D-C4       RHO=  7.7577106D-04       MAGNITUDE=  1.8096461D 00       PHASE= 2

      F= -1.0005812D 03     P= -1.0004438D 03     G= -1.0007185D 03     RSIGMA=  1.3733936D-01    H=  0.0

     THE CURRENT VALUE OF X IS
     6.9998568D 03       4.5643866D 03       8.4218738D C3       1.5712984D 00       1.7937143D 00       1.1248035D 00
     1.2242877D 00       9.2240364D-01
     THE CONSTRAINT VALUES
                                  NOT INCLUDING THE NON-NEGATIVITIES
      1.6350773D-01       1.3868661D-01       1.5095115D-01       1.4322202D-01       1.3067106D-01       1.2423937D-01
      7.7809781D-03       2.4287016D 00       2.2062857D 00       2.8751965D 00       2.7757123D 00       3.0775964D 00

   2ND  ORDER ESTIMATES

      F= -1.0007207D 03     P= -1.0007245D 03     G= -1.0007144D 03     RSIGMA=  0.0              H=  0.0

     THE CURRENT VALUE OF X IS
     7.0000052D 03       4.5656385D 03       8.4233053D C3       1.5714241D 00       1.7935165D 00       1.1247863D 00
     1.2232174D 00       9.2133852D-01
     THE CONSTRAINT VALUES
                                  NOT INCLUDING THE NON-NEGATIVITIES
      3.9355305D-03      -9.5589533D-05       2.7528389D-03      -5.1606359D-03      -1.7343593D-02      -1.3791982D-02
      3.5814299D-04       2.4285759D C0       2.2064835D 00       2.8752137D 00       2.7767826D 00       3.0786615D 00

   1ST  ORDER ESTIMATES

      F= -1.0007206D 03     P= -1.0007525D 03     G= -1.0007144D 03     RSIGMA=  0.0              H=  0.0

     THE CURRENT VALUE OF X IS
```

```
     7.0000052D 03        4.5656477D 03        8.4232941D 03        1.5714235D 00        1.7935122D 00        1.1247867D 00
     1.2232777D 00        9.2134598D-01
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      4.5734542D-03        8.4549551D-04        3.5469834D-03       -5.2355519D-03       -1.6260403D-02       -1.2231445D-02
      3.4952830D-04        2.4285755D 00        2.2064878D 00        2.8752133D 00        2.7767923D 00        3.0786540D 00


 LAGRANGE MULTIPLIERS

       F= -1.0007206D 03     P= -1.0007525D 03     G= -1.0007144D 03     RSIGMA=  0.0                H=  0.0

     THE CURRENT VALUE OF X IS
     1.5832715D-11        3.7236489D-11        1.0937451D-11        3.1420727D-04        2.4111656D-04        6.1316912D-04
     5.1755765D-04        9.1178313D-04
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      2.9317299D-02        4.0333372D-02        3.4045577D-02        3.7819344D-02        4.5433349D-02        5.0259147D-02
      1.2813424D 01        1.3151602D-04        1.5937126D-04        9.3842270D-05        1.0068961D-04        8.1904973D-05
 APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

****************************************************

        POINT=1922        DOTT=  5.7203041D-05      RHO=  1.9394276D-05      MAGNITUDE=  2.1442092D 00      PHASE= 2

       F= -1.0007009D 03     P= -1.0006804D 03     G= -1.0007214D 03     RSIGMA=  2.0501115D-02      H=  0.0

     THE CURRENT VALUE OF X IS
     6.9999777D 03        4.5655757D 03        8.4229951D 03        1.5714071D 00        1.7935038D 00        1.1247981D 00
     1.2232456D 00        9.2156265D-01
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      2.6933483D-02        2.2373643D-02        2.4656731D-02        2.2236742D-02        1.9420678D-02        1.8772593D-02
      1.2934763D-03        2.4285929D 00        2.2064962D 00        2.8752019D 00        2.7767542D 00        3.0784373D 00


   2ND  ORDER ESTIMATES

       F= -1.0007235D 03     P= -1.0007242D 03     G= -1.0007215D 03     RSIGMA=  0.0                H=  0.0

     THE CURRENT VALUE OF X IS
     7.0000003D 03        4.5658029D 03        8.4232035D 03        1.5714276D 00        1.7934630D 00        1.1247973D 00
     1.2230460D 00        9.2140621D-01
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      1.1617276D-03        5.2951940D-04        8.7078925D-04       -3.0268312D-04       -1.0896755D-03       -7.2792656D-04
      5.6541380D-05        2.4285724D 00        2.2065370D 00        2.8752027D 00        2.7769540D 00        3.0785938D 00


   1ST  ORDER ESTIMATES

       F= -1.0007234D 03     P= -1.0007249D 03     G= -1.0007215D 03     RSIGMA=  0.0                H=  0.0

     THE CURRENT VALUE OF X IS
     7.0000604D 03        4.5657990D 03        8.4232057D 03        1.5714275D 00        1.7934642D 00        1.1247971D 00
     1.2230591D 00        9.2140471D-01
     THE CONSTRAINT VALUES
                          NOT INCLUDING THE NON-NEGATIVITIES
      1.2475207D-03        5.3739090D-04        9.3769057D-04       -4.2600484D-04       -1.4689407D-03       -1.0154932D-03
      6.3874243D-05        2.4285725D 00        2.2065358D 00        2.8752029D 00        2.7769499D 00        3.0785953D 00
```

```
LAGRANGE MULTIPLIERS

     F= -1.00072340 03     P= -1.00072490 03     G= -1.00072150 03     RSIGMA=  0.0               H=  0.0

    THE CURRENT VALUE OF X IS
    3.95604060-13         9.30427390-13         2.73363470-13         7.85409500-06         6.02932910-06         1.53293750-05
    1.29612250-05         2.28362010-05
    THE CONSTRAINT VALUES
                                NOT INCLUDING THE NON-NEGATIVITIES
     2.67951720-02         3.87436020-02         3.19008740-02         3.90463470-02         5.14215070-02         5.50332350-02
     1.17732900 01         3.29824430-06         3.98352190-06         2.34604790-06         2.51535160-06         2.04650560-06
```

11. APPENDIX C: GEOMETRIC PROGRAMMING, COMPUTER OUTPUT

```
        PROBLEM HAS  7 VARIABLES  7 INEQUALITY AND  0 EQUALITY CONSTRAINTS
        AND  3 DEGREES OF DIFFICULTY
HERE COMES THE INPUT DATA CC(I,J) GROUPED BY 0 TO M

 1.000000000000000E+00

 1.000000000000000E+00   1.000000000000000E+00

 1.000000000000000E+00

 9.999999999999999E-02   9.999999999999999E-02

 1.500000000000000E+00

 1.000000000000000E+00   1.000000000000000E+00

 9.999999999999999E-02

 9.999999999999999E-02
HERE COMES THE INPUT DATA A(I,J,K) GROUPED BY VAR.

-5.999999999999999E-01   1.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00  -5.999999999999999E-01
 5.999999999999999E-01

-3.999999999999999E-01   0.000000000000000E+00   0.000000000000000E+00   1.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00  -3.999999999999999E-01
 3.999999999999999E-01

 0.000000000000000E+00   0.000000000000000E+00   1.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00  -1.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00

 0.000000000000000E+00  -7.999999999999998E-01  -7.999999999999998E-01   0.000000000000000E+00   1.000000000000000E+00
 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00

 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00  -8.999999999999998E-01   0.000000000000000E+00
 1.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00

 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00  -1.000000000000000E+00   0.000000000000000E+00
 1.000000000000000E+00   0.000000000000000E+00   1.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00

 0.000000000000000E+00  -1.000000000000000E+00  -1.000000000000000E+00   0.000000000000000E+00   1.000000000000000E+00
 0.000000000000000E+00   0.000000000000000E+00   0.000000000000000E+00   1.000000000000000E+00   0.000000000000000E+00
 0.000000000000000E+00

        PRIMAL VARIABLE STARTING POINT

             X( 1)=      0.10000000

             X( 2)=      2.00000000

             X( 3)=      2.40000000

             X( 4)=     10.00000000

             X( 5)=      5.00000000
```

```
     X( 6)=      0.50000000

     X( 7)=      0.50000000

INITIAL VALUE OF PRIMAL OBJECTIVE FUNCTION      3.01708817E+00

INITIAL VALUE OF PRIMAL CONSTRAINTS
     G( 0)=      3.01708817
     G( 1)=      0.79244660
     G( 2)=      0.93969515
     G( 3)=      0.75000000
     G( 4)=      0.62500000
     G( 5)=      1.00000000
     G( 6)=      0.30170882
     G( 7)=      0.03314454

INITIAL VALUES OF DUAL MULTIPLIERS
     TAU( 0) =          -0.10429218
     TAU( 1) =          -2.30258509
     TAU( 2) =           0.69314718
     TAU( 3) =           0.87546874
     TAU( 4) =           2.30258509
     TAU( 5) =           1.60943791
     TAU( 6) =          -0.69314718
     TAU( 7) =          -0.69314718
     TAU( 8) =           1.00000000
     TAU( 9) =           1.00000000
     TAU(10) =           1.00000000
     TAU(11) =           1.00000000
     TAU(12) =           1.00000000
     TAU(13) =           1.00000000
     TAU(14) =           1.00000000
```

```
ITERATION NUMBER     1
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          3.01708817
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.03944118

ITERATION NUMBER     2
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          3.01708817
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.45251498

ITERATION NUMBER     3
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          1.74315055
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.44241147

ITERATION NUMBER     4
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.89006593
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.39348441

ITERATION NUMBER     5
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.62275919
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.24337966

ITERATION NUMBER     6
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.56240067
APPROX. VALUE OF DUAL OBJECT. FUNCT.            1.03336320

ITERATION NUMBER     7
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.40845325
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.55947341

ITERATION NUMBER     8
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36629719
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.38789633

ITERATION NUMBER     9
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36745176
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36517971

ITERATION NUMBER    10
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773917
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36742131

ITERATION NUMBER    11
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773076
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36772666

ITERATION NUMBER    12
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36772619

ITERATION NUMBER    13
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36772905

ITERATION NUMBER    14
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36773011

ITERATION NUMBER    15
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
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APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36773051

ITERATION NUMBER   16
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36773065

ITERATION NUMBER   17
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36773070

ITERATION NUMBER   18
APPROX. VALUE OF PRIMAL OBJECT. FUNCT.          0.36773073
APPROX. VALUE OF DUAL OBJECT. FUNCT.            0.36773072

CONSTRAINTS TIGHT - OPTIMAL SOLN
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DUAL CONSTRAINT INFEASIBILITY
        0          -2.22044605E-16
        1           1.84845548E-08
        2           1.23230365E-08
        3           1.38777878E-16
        4          -2.63677968E-16
        5          -8.32667268E-17
        6          -7.54604712E-17
        7          -3.19189120E-16
        8           2.49800181E-16
        9           0.00000000E+00
       10          -2.22044605E-16
       11           1.38777878E-17
       12          -2.77555756E-17
       13          -6.07553430E-08
       14          -8.86912807E-08


OPTIMAL VALUES OF DUAL VARIABLES

A. VARIABLES  FOR PRIMAL OBJECT. FUNCT.
        1           1.00000000E+00

B. VARIABLES FOR INEQUAL CONSTS.

            INEQUALITY CONSTRAINT NUMBER  1
        1           6.00000000E-01
        2           3.10586583E-01
   DEL( 1,0)=       9.10586583E-01

            INEQUALITY CONSTRAINT NUMBER  2
        1           4.00000000E-01
   DEL( 2,0)=       4.00000000E-01

            INEQUALITY CONSTRAINT NUMBER  3
        1           7.28469266E-01
        2           3.60000000E-01
   DEL( 3,0)=       1.08846927E+00

            INEQUALITY CONSTRAINT NUMBER  4
        1           3.10586583E-01
   DEL( 4,0)=       3.10586583E-01

            INEQUALITY CONSTRAINT NUMBER  5
        1           4.00000000E-02
        2           1.82117317E-01
   DEL( 5,0)=       2.22117317E-01

            INEQUALITY CONSTRAINT NUMBER  6
        1           2.31945414E-09
   DEL( 6,0)=       6.30747972E-08

            INEQUALITY CONSTRAINT NUMBER  7
        1           3.31270451E-08
   DEL( 7,0)=       1.21818326E-07
```

```
VALUES OF DUAL MULTIPLIERS
     TAU( 0)=      2.00040431E+00
     TAU( 1)=      1.06393205E+00
     TAU( 2)=      9.05112707E-01
     TAU( 3)=      4.05465108E-01
     TAU( 4)=      2.09955750E+00
     TAU( 5)=      2.91048772E+00
     TAU( 6)=     -1.71432624E+00
     TAU( 7)=     -1.98554619E-01
     TAU( 8)=     -9.06333710E-01
     TAU( 9)=     -8.37092681E-02
     TAU(10)=     -1.08477237E+00
     TAU(11)=      1.69292566E-01
     TAU(12)=      5.04549584E-01
     TAU(13)=      1.55789446E+01
     TAU(14)=      1.49207350E+01


VALUES OF OPTIMAL PRIMAL VARIABLES
     X( 1) =         2.89774269
     X( 2) =         2.47221054
     X( 3) =         1.50000000
     X( 4) =         8.16255722
     X( 5) =        18.36575373
     X( 6) =         0.18008501
     X( 7) =         0.81991499


OPTIMAL VALUE OF PRIMAL OBJECTIVE FUNCTION       3.67730733E-01


OPTIMAL VALUE OF DUAL OBJECTIVE FUNCTION         3.67730721E-01
```